# Distributed Stability Analysis and Control of Dynamic Networks

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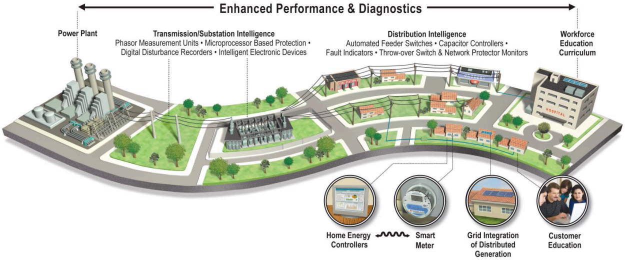
# Analysis and control of complex systems

## Paradigmatic Example

Power grids comprised of thousands nonlinear components linked across thousands of kilometers.




## Problems

The grid aging, centralized, and inflexible control infrastructure does not provide the flexibility which power systems need to guarantee its robust stability.

## Paradigm Shift

The new demands on the electricity delivery system are requiring that it function in ways for which it was never originally designed.

## Challenges (Opportunities)

- Distributed & renewable generation, emerging demand response, real time measurements (PMU).
- Stochastic and epistemic uncertainty.
- The system's dynamic components become increasingly independent and autonomous agents.

## Future Transformations

Modern distributed measurements, control, and management concepts are necessary to operate the grid.

## Project Goals

- Construct an algorithmic synthesis of nonlinear and distributed control techniques for large networked (power) systems.
- Build a decentralized approach which exploits separability and decomposition of the corresponding control problem into nonlinear sub-problems which can be solved locally and efficiently.
- Design scalable analysis and control algorithms.

# The problem we study

## Basic Stability Problem

We assume a dynamical system described by an autonomous set of nonliner equations

$$\dot{x} = f(x)\,, \quad x \in \mathbb{R}^m\,. \tag{1}$$

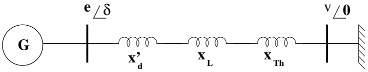
Assume that $x = 0$ is a stable equilibrium point (SEP), i.e. $f(0) = 0$.

## Fundamental Stability Question

Assume that the system reaches the state $x_c$ when the disturbance is finally cleared. Does the trajectory $x(t)$ with $x(0) = x_c$ converge to the SEP as $t \to \infty$?

# Power System Example I

- Consider this model:




$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 10\lambda - 20\sin(x_1) - x_2$$

- The equilibrium points can be found from the steady-state (power flow) equations:

$$0 = x_2$$
$$0 = 10\lambda - 20\sin(x_{10}) - x_{20}$$

Milano, F., *Power System Modelling and Scripting*, Springer, Heidelberg.

# Equilibria

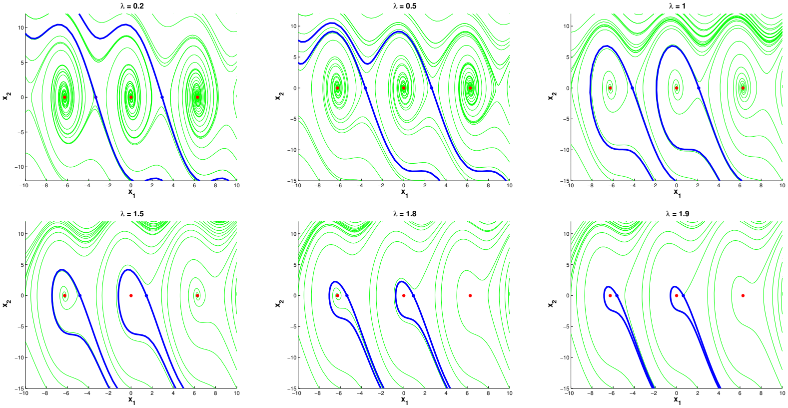
- The solutions are:

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} \sin^{-1}(\lambda/2) \\ 0 \end{bmatrix} \tag{2}$$

- With two equilibrium points (and their periodic images):

$$x_{1s} = \sin^{-1}(\lambda/2)$$
$$x_{1u} = \pi - \sin^{-1}(\lambda/2)$$

# Stability and Region of Attraction



# Power System Example II

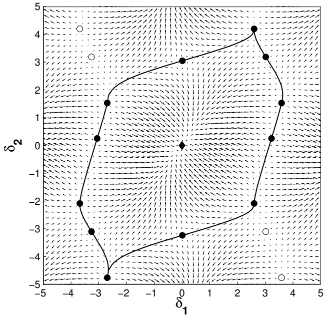
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin(x_1) - 0.5\sin(x_1 - x_3) - 0.4x_2$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -0.5\sin(x_3) - 0.5\sin(x_3 - x_1) - 0.5x_4 + 0.05$$

where $x_1 = \delta_1, x_2 = \omega_1, x_3 = \delta_2, x_4 = \omega_2$.

**Remark**

No transfer conductances!

Jump to Energy Function | Jump to Classical Model




The boundary of the region of attraction for the SEP $x_s$ located at the origin (◇). This boundary contains 12 hyperbolic equilibrium points (●). Four more equlibrium points are also shown (○).

H. D. Chiang, *Direct Methods for Stability Analysis of Electric Power Systems*,Wiley, 2011.

## Lyapunov's Theorem

If $\exists \mathcal{D} \subset \mathbb{R}^m$ containing $x_s = 0$ and $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0\,, \quad \text{and}\, V(x) > 0 \quad \forall x \in \mathcal{D} \setminus \{0\}\,,$$
$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \forall x \in \mathcal{D}\,,$$

then the origin is a stable equilibrium. Moreover, if

$$\dot{V}(x) < 0 \quad \forall x \in \mathcal{D} \setminus \{0\}\,,$$

then the origin is asymptotically stable.

## ROA estimates

Any level set $\Omega_c = \{x \in \mathbb{R}^m \mid V(x) \leq c\}$, such that $\Omega_c \in \mathcal{D}$, describes a positively invariant region contained in the domain of attraction of the equilibrium point.

# The trouble with Lyapunov

- They are not constructive conditions; they do not tell us how to find a Lyapunov function for a particular system.
- Testing the positivity conditions required in the theorem is notoriously difficult.
- Even when both the vector field $f$ and the Lyapunov function candidate $V$ are polynomial, the Lyapunov conditions are essentially polynomial non-negativity conditions which are known to be $\mathcal{NP}$-hard to test.
- Fortunately, if we relax the polynomial non-negativity conditions to appropriate sum of squares (SOS) conditions, testing SOS conditions can be done efficiently using SDP.

# Positive Polynomials

## Gramm Matrix Representation

Any $p \in \mathcal{R}_{m,2d}$ can be represented as $p(x) = z_{m,d}(x)^T Q z_{m,d}(x)$, where $z_{n,d}(x)$ is a vector of monomials

$$z_{m,d}(x) := [1, x_1, x_2, \ldots, x_m, x_1^2, x_1 x_2, \ldots, x_m^2, \ldots, x_m^d]^T \qquad (3)$$

Remark: $z_{m,d}(x)$ is a $\binom{m+d}{d}$ -vector.

## SOS polynomials

$p$ is a sum of squares (SOS) if there exist polynomials $\{p_i\}_{i=1}^N$ such that $p = \sum_{i=1}^N p_i^2$. If $p$ is SOS then $p \geq 0$. The reverse implications is not true except in some specific cases.

## Theorem

*Fix $p \in \mathcal{R}_{m,2d}$. Then $p(x) \in \Sigma_{m,2d}$ iff there exists $Q \succeq 0$.*

## Remark

- All solutions to $p(x) = z_{m,d}(x)^T Q z_{m,d}(x)$ can be expressed as $Q = Q_0 + \sum_{i=1}^{n} \lambda_i Q_i$ where $p = z_{m,d}^T Q_0 z_{m,d}$ and each $Q_i$ satisfies $z_{m,d}^T Q_i z_{m,d} = 0$.
  Jump to SOS example
- Determining if a SOS decomposition exists for a given polynomial is equivalent to a linear matrix inequality feasibility problem.

$$\begin{aligned} \text{find} \quad & \lambda_1, \ldots, \lambda_n \\ \text{s.t.} \quad & Q_0 + \sum_{i=1}^{n} \lambda_i Q_i \succeq 0 \end{aligned} \tag{4}$$

# SOS programming

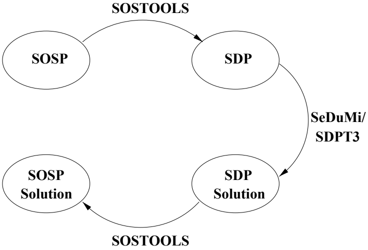
- Given $w \in \mathbb{R}^s$ and polynomials $\{p_k\}_{k=0}^s$ solve:

$$\min_{\alpha \in \mathbb{R}^m} \quad w^T \alpha$$

$$\text{subject to:} \quad p_0 + \sum_{k=1}^{s} \alpha_k p_k \in \Sigma[x]$$

- This SOS programming problem is an SDP:
  - The cost is a linear function of $\alpha$.
  - The SOS constraint can be replaced with a LMI constraint.

# SOSTOOLS




- Converts the SOS program to a SDP.
- Calls the SDP solver.
- Converts the SDP solution back to the solution of the SOS program.

S.Prajna, A.Papachristodoulou, and P.A.Parrilo. *SOSTOOLS — A Sum of Squares Optimization Toolbox*, 2013.

http://www.cds.caltech.edu/sostools/

## SOS search for Lyapunov functions

SOS technique is used to find polynomial functions $V \in \mathcal{R}_m$, with $V(0) = 0$ such that

$$V(x) - \phi_1(x) \in \Sigma_m \forall x \in \mathcal{D} \quad (5)$$

$$-\dot{V}(x) - \phi_2(x) \in \Sigma_m \forall x \in \mathcal{D} \quad (6)$$

for some domain $\mathcal{D}$ around $x = 0$ and positive definite functions $\phi_{1,2}(x)$. For example $\phi_{1,2}(x) = \epsilon_{1,2} \sum_{i=1}^{m} x_i^2$ and $\epsilon_{1,2} > 0$.

## Notes

It is convenient to define the domain $\mathcal{D} = \{x \in \mathbb{R}^m \mid p(x) \leq \beta\}$ for some positive polynomial $p(x) = \sum_{i=1}^{m} x_i^2$ and $\beta > 0$.

# Representation Theorems

## Putinar's Positivestellensatz Theorem

Let $\mathcal{K} = \{x \in \mathbb{R}^m \mid g_1(x) \geq 0, \dots, g_n(x) \geq 0\}$ be a compact set. If $p(x)$ is positive on $\mathcal{K}$, then $p(x) = \sigma_0 + \sum_j \sigma_j g_j(x) \mid \sigma_0, \sigma_j \in \Sigma_m, \forall j$.

## SOS problem

Given $p(x)$ find $\sigma_j \in \Sigma_m, j = 1, 2, \dots, n$ such that $p(x) - \sum_j \sigma_j g_j(x) \in \Sigma_m$. We can also search for unknown coefficients of $p$ so that $p$ is positive on $\mathcal{K}$.

M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana University Mathematics Journal, vol. 42, no. 3, pp. 969 - 984, 1993.
J. B. Lasserre, *Moments, Positive Polynomials and Their Applications*, Imperial College Press, 2010.

# Power System Example

## SOS search for Lyapunov functions

If there exists a constant $\beta > 0$ and polynomial functions $V, p_{1,2} \in \mathcal{R}_m$, and $\sigma_{1,2} \in \Sigma_m$ such that $V(0) = 0$ and

$$V - \sigma_1(\beta - p) - p_1 G - \phi_1 \in \Sigma_m \quad (7)$$

$$-\dot{V} - \sigma_2(\beta - p) - p_2 G - \phi_2 \in \Sigma_m \quad (8)$$

then $x = 0$ is a stable equilibrium point.

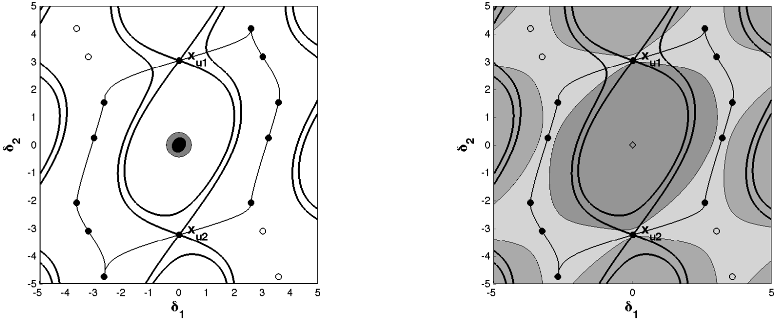
## Remark

The vectors of polynomials $G$ enforce algebraic constraints produced by a recasting procedure that transforms the non-polynomial ODEs into a set of polynomials DAEs.

Jump to Analysis of Non-polynomial systems

## Remark

An algorithm to maximize the size of the invariant subset is needed in order to improve the estimated ROA.





Wloszek, *Lyapunov based analysis and controller synthesis for polynomial systems using SOS optimization*, Ph.D. dissertation, UC Berkeley, 2003.
Anghel et al, *Algorithmic Construction of Lyapunov Functions for Power System Stability Analysis*. TCS 60, 2533-2546, 2013.


Go to Bretas model

# The Trouble with SOS

- There are serious difficulties before these algebraic methods can be applied to large power systems.
- The difficulties are not conceptual but numerical.
- It is currently very hard to construct Lyapunov functions of systems with large state dimension, for cubic vector fields and quartic Lyapunov functions.
- This limitation renders the proposed algorithm impractical in its current formulation.

## Decomposition/Aggregation Approach

System decomposition and analysis of its components (subsystems) and their interconnections.

J. Anderson and A. Papachristodoulou, *A decomposition technique for nonlinear dynamical system analysis*, IEEE TAC 57, 1516 - 1521, June 2012.

## Fundamental Questions

- How to guarantee the global stability of a nonlinear interconnected system form the local stability of its subsystems?
- How to design local controllers that respond to local disturbances?

## Generic Analysis Tools

1. vector Lyapunov functions and linear comparison principles
2. composite Lyapunov functions (do not scale well!)
3. small-gain theorems for networked systems (do not scale well!)

## Proposed Approach

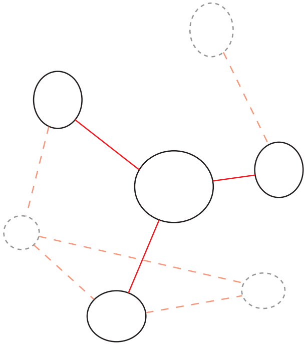
1. We employ a system decomposition technique in order to derive a collection of low order, weakly interacting subsystems.
2. We perform a Lyapunov stability analysis for each isolated subsystem.
3. We analyze the stability of the full system using the subsystem Lyapunov functions and disturbance analysis techniques.

# Network of Dynamical systems

We seek a decomposition

$$\dot{x}_1 = f_1(x_1) + g_1(x_1, x_{\mathcal{N}_1})$$
$$\dot{x}_2 = f_2(x_2) + g_2(x_2, x_{\mathcal{N}_2})$$
$$\vdots$$
$$\dot{x}_n = f_n(x_n) + g_n(x_n, x_{\mathcal{N}_n})$$

where $f_i(0) = 0$, $g_i(x_i, 0) = 0, \forall i$, and $\mathcal{N}_i$ : {neighbors of 'i'}.



## Fundamental Questions

- Is it asymptotically stable, i.e. $\lim_{t\to\infty} x_i(t) = 0$?
- If not, is there a distributed stabilizing control $u_i(x_i)$?

# Isolated Subsystem Analysis

Analyze the isolated dynamics $\dot{x}_i = f_i(x_i)$

- If $\exists\, \mathcal{D}_i \subset \mathbb{R}^{m_i}$ and $V_i : \mathcal{D}_i \to \mathbb{R}$ such that

$$V_i(0) = 0\,, \quad V_i(x_i) > 0 \quad \forall x_i \in \mathcal{D}_i \setminus \{0\}\,,$$
$$\dot{V}_i(x_i) = \nabla V_i^T f_i(x_i) < 0 \quad \forall x_i \in \mathcal{D}_i\,,$$

- Finaly, $\mathcal{R}_i^0 = \{x_i \in \mathbb{R}^{m_i} \mid V_i(x_i) \leq 1\} \subseteq \mathcal{D}_i$ ,

## Let's point the obvious

- With no interactions $\mathcal{R} = \mathcal{R}_1^0 \times \mathcal{R}_2^0 \times \ldots \times \mathcal{R}_n^0$ approximates the global ROA.
- Under interactions from neighbors $\dot{V}_i(x_i) = \nabla V_i^T (f_i(x_i) + g_i) < 0$ only if $\|g_i\|_2$ is sufficiently small. This is not quite so as $x_i \to 0$.

## Stability of Two Connected Systems

- Assume two interacting subsystems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1, x_2)\,, \\ \dot{x}_2 &= f_2(x_2) + g_2(x_2, x_1)\,. \end{aligned}$$

- We assume that $f_i(x_i) = 0$ and $g_i(x_1, x_2) = 0$, for $i = 1, 2$.
- There $\exists$ Lyapunov functions $V_i(x_i)$ for each isolated subsystem $\dot{x}_i = f_i(x_i)$, and ROA estimates given by $\Omega_1^i = \{x_i \in \mathbb{R}^{n_i} \mid V_i(x_i) \le 1\}$ for $i = 1, 2$.

# Example I

## Coupled Van der Pol oscillators

- Dynamics:

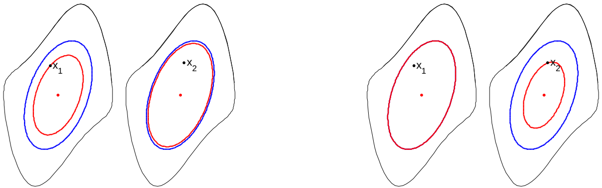
$$\begin{aligned}
\dot{x}_1 &= -x_2 + a x_2 x_3 \\
\dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 + b x_1 x_4 \\
\dot{x}_3 &= -x_4 + a x_4 x_1 \\
\dot{x}_4 &= x_3 + (x_3^2 - 1)x_4 + b x_3 x_2 \,,
\end{aligned}$$

- There $\exists$ quadratic Lyapunov functions :

$$\begin{aligned}
V_1 &= 0.77 x_1^2 - 0.49 x_1 x_2 + 0.47 x_2^2 \,, \\
V_2 &= 0.77 x_3^2 - 0.49 x_3 x_4 + 0.47 x_4^2 \,.
\end{aligned}$$

## Problem Formulation

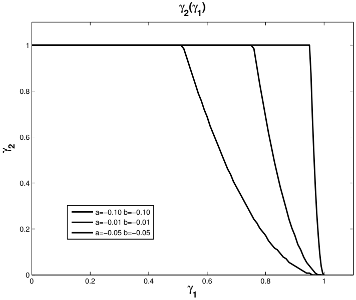
- Assume that a disturbance moves the coupled system to the state $(x_1^0, x_2^0)$ after which the disturbance is removed.
- Will the system evolve back to its stable equilibrium $(0, 0)$?




## Compute Stability Curves

Find the $\gamma_2(\gamma_1)$ such that $\Omega^1_{\gamma_1}$, $\gamma_1 \leq 1$, remains invariant under bounded disturbances $\Omega^2_{\gamma_2}$, $\gamma_2 \leq 1$, and the action of the system's dynamics.

# Example I




## Stability Condition

For two connected systems the answer is very simple:

- If for $\gamma_1 = V_1(x_1^0)$ the state $x_2^0 \in \{x_2 \in \mathbb{R}^n \mid V_2(x_2) \leq \gamma_2(\gamma_1)\}$,
- If for $\gamma_2 = V_2(x_2^0)$ the state $x_1^0 \in \{x_1 \in \mathbb{R}^n \mid V_1(x_1) \leq \gamma_1(\gamma_2)\}$,

then the composite system is stable.

## Network Stability Conditions

- Let's assume a system composed of $n$ subsystems:

$$\dot{x}_i = f_i(x_i) + \sum_{j \in \mathcal{N}(i)} g_{ij}(x_i, x_j)\,, \text{for} \quad k = 1, \ldots, M. \tag{9}$$

- Assume that the system is in the state $(x_1^0, \ldots, x_n^0)$.
- Compute $\gamma_i = V_i(x_i^0)$. For each $i$ solve the SOS programs

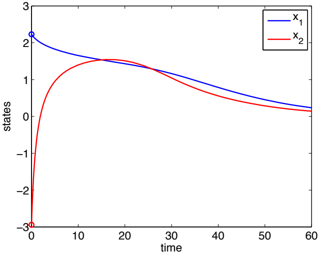
$$-s_2 \dot{V}_i - \sum_{j \in \mathcal{N}(i)} s_{1j}(\gamma_j - V_j(x_j)) - q(V_i - \gamma_i) \in \Sigma_{n+m}\,. \tag{10}$$

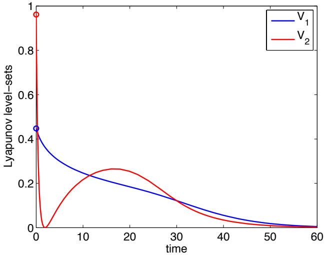
- If a solution exists for all $i$ the system is stable.
- The subsystems for which a solution is not found might be unstable. Apply local controllers to restore stability.

# Example II

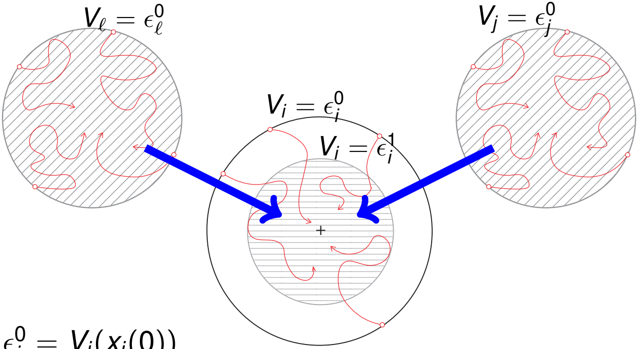
## Coupled 1D Systems

$$\dot{x}_1 = -0.1x_1 + 0.03x_1^2 {\color{red}+0.04x_2}, \quad V_1 = 0.09x_1^2, \mathcal{R}_1^0 = [-3.3, 3.3].$$
$$\dot{x}_2 = -0.6x_2 + 0.20x_2^2 {\color{red}+0.30x_1}, \quad V_2 = 0.11x_2^2, \mathcal{R}_1^0 = [-3.0, 3.0].$$



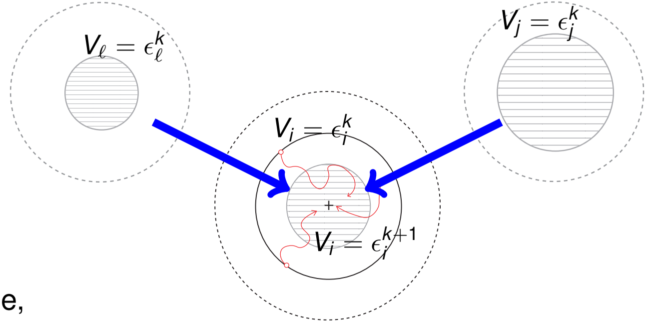



# Iterative Stability Algorithm: Step # 1




- Communicate $\epsilon_i^0 = V_i(x_i(0))$
- Then, for each $i$, find the minimum $\epsilon_i^1 \in [0, \epsilon_i^0)$, such that $\nabla V_i^T(f_i + g_i) < 0$ on $\{x \,\big|\, V_i \in [\epsilon_i^1, \epsilon_i^0], V_j \in [0, \epsilon_j^0] \,\forall j \in \mathcal{N}_i\}$

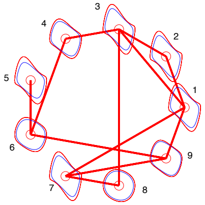
# Iterative Stability Algorithm: Step # k




If $\epsilon_i^k$=0 : STOP. Else,

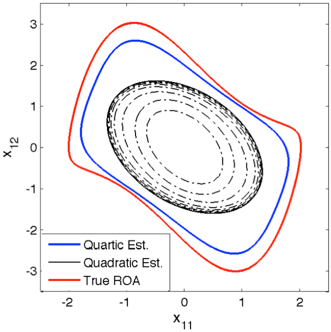
- Communicate $\epsilon_i^k$ (from step k)
- Then, for each $i$, find the minimum $\epsilon_i^{k+1} \in [0, \epsilon_i^k)$, such that $\nabla V_i^T(f_i + g_i) < 0$ on $\{x \,\middle|\, V_i \in [\epsilon_i^{k+1}, \epsilon_i^k], V_j \in [0, \epsilon_j^k]\, \forall j \in \mathcal{N}_i\}$

# Example I: Network of Van Der Pol Oscillators

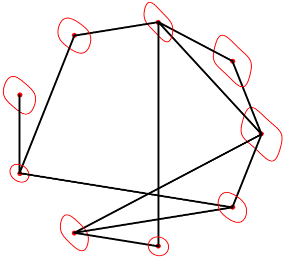



$$\dot{x}_{i1} = x_{i2}$$
$$\dot{x}_{i2} = \mu_i \, x_{i2} \, (1 - x_{i1}^2) - x_{i1} + \sum_{j \in \mathcal{N}_i \setminus \{i\}} \zeta_{ij} x_{i1} x_{j2}$$

- Polynomial Lyapunov functions, using
- *Expanding interior algorithm*, see Wloszek 2003, Anghel *et al* 2013, and
- Estimates of isolated ROAs for varying polynomial orders.



# Example I: Network of Van Der Pol Oscillators



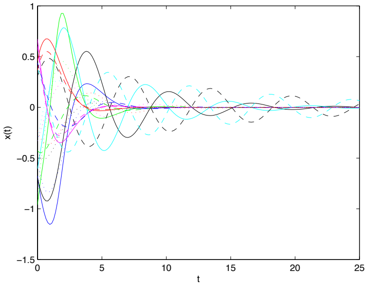
The red lines are the contour level sets defined by the initial conditions: $\epsilon_i^0 = V_i(x_i(0))$.

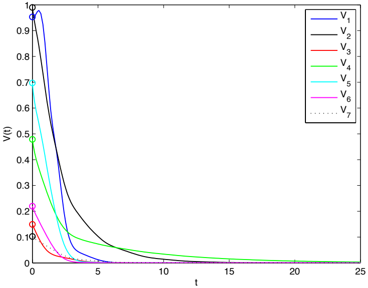
Evolution of the stability analysis algorithm.

# Stabilizing Control Design

- Necessary when the stability algorithm fails
- Attributes:
  - *Decentralized:* activated and designed locally
  - *Minimal:* applied on certain "rings" in the state-space
- Design method (at each step k, for each i),
  - Decide if control is needed,
    i.e. if $\nabla V_i^T (f_i + g_i)\big|_{V_i=\epsilon_i^k,\, V_j \leq \epsilon_j^k\, \forall j \neq i} \geq 0$ .
  - If needed, compute control $u_i^k(x_i)$, so that
    $\nabla V_i^T (f_i + g_i + u_i^k)\big|_{V_i=\epsilon_i^k,\, V_j \leq \epsilon_j^k\, \forall j \neq i} < 0$
  - Find the minimum $\epsilon_i^{k+1} \in [0, \epsilon_i^k)$ with the controlled dynamics.

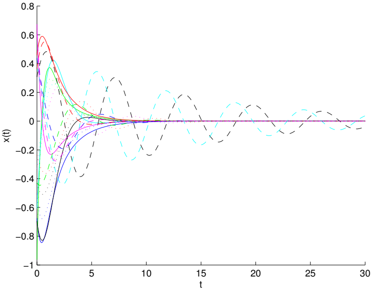
# Control Example

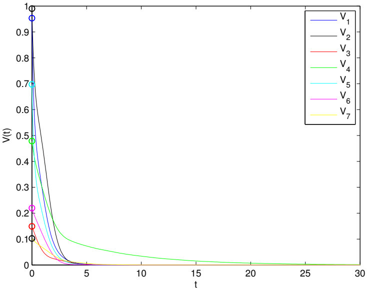



- Control applied at subsystems – 1, 2 and 5.
- Conservative design, due to (1) its distributed nature, and (2) the inherent conservativeness of the Lyapunov functions

# Control Example




- Control applied at subsystems – 1, 2 and 5.
- Conservative design, due to (1) its distributed nature, and (2) the inherent conservativeness of the Lyapunov functions

# Example II: Power System Example

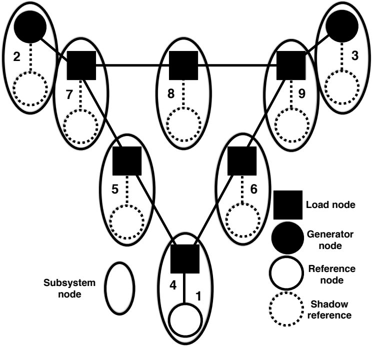
- Network preserving model:

$$D_i\dot{\delta}_i = -P_{D_i} - P_{E_i}(\delta)\,,\ i = 1,\ldots,L\,,$$
$$\dot{\delta}_{L+i} = \omega_i\,,\ i = 1,\ldots,G\,,$$
$$M_i\dot{\omega}_i + D_{L+i}\omega_i = P_{M_i} - P_{E_{L+i}}(\delta)\,.$$

Bergen & Hill, *A structure preserving model for power system stability analysis*, TPAS, 100, 25-35, 1981.

- Overlapping decomposition:

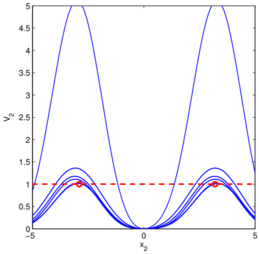
$$\delta_{in} = \delta_i - \delta_n, \text{ for } i = 1,\ldots,L+G-1,$$
$$\omega_{in} = \omega_i - \omega_n, \text{ for } i = 1,\ldots,G-1,$$
$$\omega_G = \omega_n.$$



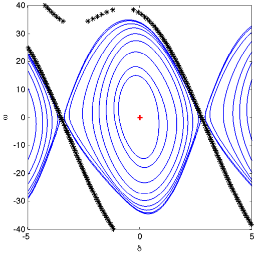

Network of three generators and six load nodes. We perform an overlapping decomposition in which the speed dynamics of the reference node (generator node 3) is shared with all the subsystems.

Jump to Dynamic Load Model

# Example II: Power System Example
## Disturbance Analysis




(a) ROA of subsystem 2 (node 5)




(b) ROA of subsystem 7 (node 2)

Estimates of regions of attraction of isolated subsystems using expanding interior algorithm.

# Stability in the sense of Lyapunov, under control

A disturbance was created by tripping the line between nodes 5-7 for the duration $t \in [0,\ 3s]$ and also tripping the line between nodes 7-8 for $t \in [1s,\ 3s]$, essentially disconnecting the nodes 7 and 2 from the rest of the network for $t \in [1s,\ 2s]$.

| $k$ | $\epsilon_1^k$ | $\epsilon_2^k$ | $\epsilon_3^k$ | $\epsilon_4^k$ | $\epsilon_5^k$ | $\epsilon_6^k$ | $\epsilon_7^k$ | $\epsilon_8^k$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.0001* | 0.0007* | 0.0002* | 0.3471 | 0.0003* | 0.0001* | 0.6663* | 0.0001* |
| 1 | 0.0000 | 0.0000 | 0.0002 | 0.0312 | 0.0000 | 0.0001 | 0.4451 | 0.0000 |
| 2 | 0.0000 | 0.0000 | 0.0001 | 0.0203 | 0.0000 | 0.0000 | 0.4260 | 0.0001 |
| 3 | 0.0000 | 0.0000 | 0.0001 | 0.0195 | 0.0000 | 0.0000 | 0.3478 | 0.0000 |
| 4 | 0.0000 | 0.0000 | 0.0000 | 0.0154 | 0.0000 | 0.0000 | 0.3383 | 0.0000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 11 | 0.0000 | 0.0000 | 0.0000 | 0.0103 | 0.0000 | 0.0000 | 0.2481 | 0.0000 |
| 12 | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0000 | 0.0000 | 0.2374 | 0.0000 |
| 13 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0492 | 0.0000 |
| 14 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0094 | 0.0000 |
| 15 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

**State-Feedback Control**

State-feedback example

$$\mathcal{U}_7^0 = 5.1 \cos \delta_{2,1} - 38.4\omega_{2,1} - 1.8\omega_1 - 71.4 \sin \delta_{2,1} - 5.1$$

which is applied to the speed dynamics equation of the generator 2 of $S_7$.

## Key Points

- Distributed, scalable, and parallel algorithm for large-scale nonlinear dynamical systems
- Robust to structural changes in the network
- Local communications and measurements
- Can be extended to cases when Lyapunov function level-sets show initial increase (Kundu & Anghel 2015)
- The method opens up the possibility to include more refined subsystem models in the analysis of large (power) systems.

## Many thanks to:

- Pablo Parrilo
- Zachary W. Jarvis- Wloszek
- Antonis Papachristodoulou
- Stephen Prajna
- Weehong Tan
- Federico Milano

# Classical Power System Model

In a classical power system model consisting of $n$ synchronous generators the dynamics of the generator phase angles are modeled by the swing equations:

$$\dot{\delta}_i = \omega_i \, , \tag{11a}$$

$$\dot{\omega}_i = -\lambda_i \omega_i + \frac{1}{M_i}(P_{mi} - P_{ei}(\delta)) \, , \tag{11b}$$

where

$$P_{ei}(\delta) = E_i^2 G_{ii} + \sum_{j, j \neq i} E_i E_j [B_{ij} \sin(\delta_i - \delta_j) + G_{ij} \cos(\delta_i - \delta_j)] \, .$$

Return to Power System Example

# Energy Function

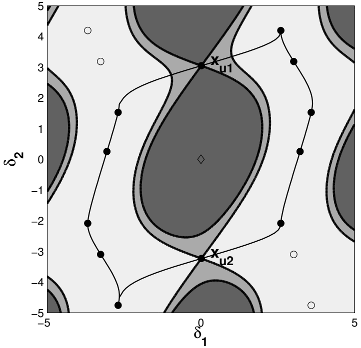
When $G_{ij} = 0$ the classical model has the following energy function:

$$V(\delta,\omega) = \frac{1}{2}\sum_i M_i\omega_i^2 - \sum_i P_i(\delta_i - \delta_i^s) - \sum_{j,j\neq i} E_iE_jB_{ij}\{\cos(\delta_i - \delta_j) - \cos(\delta_i^s - \delta_j^s)\}$$

## Estimating ROA

- The Closest UEP algorithm
- The Boundary of stability region-based Controlling Unstable equilibrium point (BCU) method.

Return to Power System Example
Return to Troubles with Lyapunov



# ... and some Remarkable Weaknesses

- No analytical energy functions for power systems with transfer conductances exists.
- The task of computing the critical energy value is very difficult.
- $\exists$ counter-examples in which these methods produce wrong results.
  A. Llamas et al, *Clarifications of the BCU method for transient stability analysis*, PES **10**,210-219,1995.
- The assumption of the method do not hold generically for power system models. There are no theoretical guarantees. F. Paganini et al, *Generic Properties, One-Parameter Deformations, and the BCU Method*, TCS **46**, 760-763, 1999.
- The only methods with zero miss probability are those who survey all UEPs. R. Fischl et al, *A comparison of dynamic security indices based on direct methods*, IJEPES **10**, 210-232,1988.

Return to Power System Example Return to Troubles with Lyapunov

# Example

- The polynomial $p = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$ can be written as $p = z_{2,2}^T Q z_{2,2}$ where

$$z_{2,2} = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix} \quad Q_0 = \begin{bmatrix} 2 & 1 & -0.5 \\ 1 & 0 & 0 \\ -0.5 & 0 & 5 \end{bmatrix} \quad Q_1 = \begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}$$

- We can define an affine subspace of symmetric matrices related to $p$ as

$$\mathcal{S}_p = \{Q | z_{n,d}^T Q z_{n,d} = p(x)\} = \left\{ Q_0 + \sum_{i=1}^{h} \lambda_i Q_i \middle| \lambda_i \in \mathbb{R} \right\}$$

- $p = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$ is SOS since $Q_0 + \lambda_1 Q_1 \succeq 0$ for $\lambda_1 = 5$.
- An SOS decomposition can be constructed from a Cholesky factorization:

$$Q + \lambda_1 Q_1 = L^T L$$

where:

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & -3 \\ 0 & 3 & 1 \end{bmatrix}$$

- Thus $p = (Lz)^T(Lz) == \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1 x_2)^2 + \frac{1}{2}(x_3^2 + 3x_1 x_2)^2$

A Tutorial on Sum of Squares Techniques for Systems Analysis Antonis Papachristodoulou and Stephen Prajna, ACC 2005.

Return to SOS Polynomials

# Analysis of Non-polynomial systems

## Theorem

Any system with non-polynomial nonlinearities can be converted to a polynomial system with a larger state dimension, but with a series of equality constraints restricting the states to a manifold of the original state dimension.

1. Define the new state variables $z_{3i-2} = \sin(\delta_i)$, $z_{3i-1} = 1 - \cos(\delta_i)$, $z_{3i} = \omega_i$ for $i = 1, \ldots, (n-1)$.
2. Use the chain rule of differentiation to derive the dynamics of the new state variables.
3. Derive the equality constraints that arise from the recasting process:

$$G_i(z) = z_{3i-2}^2 + z_{3i-1}^2 - 2z_{3i-1} = 0\,, \tag{12}$$

where $i = 1, \ldots, n-1$.

- Recasted dynamics: $\dot{z} = f(z)$, where $z = F(x)$, $f : \mathbb{R}^M \to \mathbb{R}^M$, with $M = m_1 + m_2$, and $m_1 = n - 1$ and $m_2 = 2(n-1)$.

$$
\begin{aligned}
\dot{z}_1 &= z_3 - z_2 z_3 \\
\dot{z}_2 &= z_1 z_3 \\
\dot{z}_3 &= 0.4996 z_4 - 0.4 z_3 - 1.4994 z_1 - 0.0200 z_5 + 0.0200 z_1 z_4 \\
&\quad + 0.4996 z_1 z_5 - 0.4996 z_2 z_4 + 0.0200 z_2 z_5 \\
\dot{z}_4 &= z_6 - z_5 z_6 \\
\dot{z}_5 &= z_4 z_6 \\
\dot{z}_6 &= 0.4996 z_1 + 0.0200 z_2 - 0.9986 z_4 + 0.0500 z_5 - 0.5 z_6 \\
&\quad - 0.0200 z_1 z_4 - 0.4996 z_1 z_5 + 0.4996 z_2 z_4 - 0.0200 z_2 z_5
\end{aligned}
$$

- The recasting process introduces the following constraints:

$$
\begin{aligned}
G_1(z) &= z_1^2 + z_2^2 - 2.0 z_2 = 0 \\
G_2(z) &= z_4^2 + z_5^2 - 2.0 z_5 = 0
\end{aligned}
$$

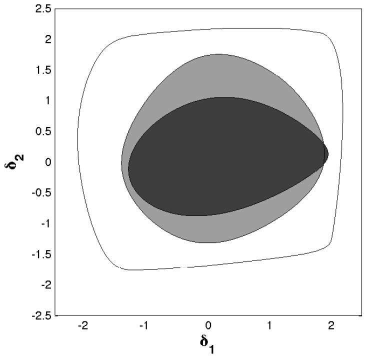
Return to Power System Example

# Another Example

## A model with transfer conductances:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 33.5849 - 1.8868\cos(x_1 - x_3) - 5.2830\cos(x_1) \\
&\quad - 16.9811\sin(x_1 - x_3) - 59.6226\sin(x_1) \\
&\quad - 1.8868x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= 11.3924\sin(x_1 - x_3) - 1.2658\cos(x_1 - x_3) \\
&\quad - 3.2278\cos(x_3) - 1.2658x_4 - 99.3671\sin(x_3) \\
&\quad + 48.4810
\end{aligned}
$$

where $x_1 = \delta_1, x_2 = \omega_1, x_3 = \delta_2, x_4 = \omega_2.$




Return to Power System Example

Bretas and Alberto, *Lyapunov function for power systems with transfer conductances: Extension of the invariance principle*, TPS 18, 769-777, 2003.

# Disturbance Analysis

## Problem Formulation

- We consider a polynomial dynamical system

$$\dot{x} = f(x) + g(x, w)\,, \tag{13}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$, $f \in \mathcal{R}_n^n$, $f(0) = 0$, and $g \in \mathcal{R}_n^{n \times m}$.
- We define the peak of $w$ to be bounded by $\|w(t)\|_\infty \leq \sqrt{\gamma_2}$.
- Define the invariant set as $\Omega_{\gamma_1} = \{x \in \mathbb{R}^n \mid V(x) \leq \gamma_1\}$, where $V(x)$, is the Lyapunov function of the isolated system ($w = 0$).
- Find the $\gamma_2$ such that $\Omega_{\gamma_1}$ remains invariant under these bounded disturbances and the action of the system's dynamics.

## SOS formulation

- If $\dot{V}(x,w) \leq 0$ on the boundary of $\Omega_{\gamma_1}$ for all $\|w(t)\|_\infty \leq \sqrt{\gamma_2}$, then $\Omega_{\gamma_1}$ is invariant.
- Solving the SOS program

$$-s_1(\gamma_2 - w^T w) - s_2\dot{V} - q(V - \gamma_1) \in \Sigma_{n+m}. \tag{14}$$

  by searching over the polynomials $q$ and the $s_i, i = 1,2$ to maximize $\gamma_2$ subject to (14).
- Compute stability curve $\gamma_2(\gamma_1)$.

## Remarks

SOS techniques can be used to perform Nonlinear Reachable Set Analysis and to incorporate Parametric Uncertainty.

Return to Stability of Two Connected Systems

# Dynamic Load Modeling

- In classical power system model, the dynamics of the system is given by the “swing equations” of the generators alone. The loads in the network are simply modeled as constant impedance for the purpose of stability analysis.
- This is not necessarily a valid assumption. Especially considering that in a “future grid” there will be increasingly higher demand side participation which is likely to introduce faster dynamics at the load bus (or node).
- One usually adopted dynamic load model represents the frequency dynamics at the load nodes as a function of their real (or active) power. However, other dynamics can also be considered, for example, the voltage magnitude dynamics as a function of the reactive load.

A. R. Bergen and D. J. Hill, *A structure preserving model for power system stability analysis*, IEEE PAS-100(1), 25-35, Jan 1981.

# Frequency-dependent Load Power

- Each load is modeled as a frequency-dependent power load, while assuming that the load bus voltage magnitude remains constant (an assumption that can be relaxed if desired).

$$\dot{\delta}_i = \frac{1}{D_i}(P_{Li} - P_{ei}(\delta)), \tag{15a}$$

where

$$P_{ei}(\delta) = E_i^2 G_{ii} + \sum_{j,j\neq i} E_i E_j [B_{ij} \sin(\delta_i - \delta_j) + G_{ij} \cos(\delta_i - \delta_j)].$$

and $P_{Li}$ is the nominal (or rated) active power load. At the limit $D_i \to 0+$, it is constant power load, with $P_{ei} = P_{Li}$.

# Frequency-dependent Load Power (Cont.)

The frequency-dependent load model has its advantages:

- In Lyapunov stability analysis, the transfer conductances in lines are often ignored, which becomes invalid if active loads are absorbed as equivalent resistances. Frequency-dependent load model resolves that issue.
- The structural aspect of the network is nicely preserved. Thus the Lyapunov function truly represents the spatial distribution of stored energy in the network (a “topological Lyapunov function”).
- This also gives us the tool to consider other dynamics, such as a voltage magnitude dependent reactive power load model.
- There are demand side control algorithms that changes the active power consumption on a real-time basis based on the network condition). Preserving the network topology helps us explicity account for such load control techniques in analyzing stability of the network.

Return to Network Preserving Model