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Power System Modelling as Stochastic Functional Hybrid Differential-Algebraic Equations

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Abstract: This paper presents the software tools developed for the research project Advanced Modelling for Power System Analysis and Simulation (AMPSAS) funded by Science Foundation Ireland from 2016 to 2021. The main objective of AMPSAS was the development of novel analytical and computational tools to understand, efficiently design, and optimise ever-changing modern power systems and smart grids, through model-based approaches. In particular, the paper discusses: (i) stochastic differential equations for modelling power systems which are subject to large stochastic perturbations (e.g., wind and solar generation); (ii) the effect of controller and modelling imperfections, e.g., delays, discontinuities, and digital signals, on both local and area-wide regulators in power systems; and (iii) the stability analysis and dynamic performance of power systems modelled through stochastic, delay and hybrid implicit differential-algebraic equations. The software tool developed during the execution of AMPSAS integrates areas of applied mathematics, automatic control, and computer science. Several implementation features and open challenges of this software tool are also discussed in the paper. A variety of examples that illustrates the features of this software tool are based on a dynamic model of the all-island Irish transmission system.

1 Introduction

1.1 Motivations

A combination of technical innovation and the increasing presence of renewable and non-conventional generation in actual electrical networks all over the world highlights the necessity of studying several aspects related to the modelling, regulation, and stability of power systems [1, 2].

Among the several factors affecting the behaviour of the electrical energy supply, it is worthwhile mentioning the volatility of some primary energy sources, such as wind and solar. This volatility is mainly due to the stochastic nature of the weather conditions that determine the wind speed and solar radiance. Moreover, the power consumed by a consumer is intrinsically affected by uncertainty. In most cases, this volatility can have a negative impact on the quality of the power supply and can reduce network security. As a matter of fact, the power fluctuations of wind power plants can potentially even help stabilise the system [3]. Proper modelling of stochastic processes can lead to surprising and counterintuitive conclusions. For these reasons, a systematic approach that properly considers stochastic models is highly desirable [4].

An immediate consequence of the presence of intermittent generators in transmission and distribution networks is the need to improve existing controllers and, in most cases, install new local and/or area regulation systems (see, for example, [5]). While local regulation is typically appropriate to maximise electrical energy production from renewable sources (e.g., the maximum power point tracker), area regulators are intended for advanced control of resources (e.g., smart grids), minimising the negative effects of the volatility previously described. The goal is that all network users, i.e., generators and consumers, can be regulated in order to guarantee the maximum cost effectiveness, quality, security, and stability of the electrical energy supply. Any control strategy, in particular with the case of area controllers, requires a communication network that can send the measured signals to control centres. These communication processes can be affected by delays, noise, discontinuities (e.g., gaps and discretisation of digital signals) and/or loss of information. In particular, delays and loss of information, just like the stochastic variations present in wind speed and solar radiance, are sources of instability and significantly increase the nonlinearity of the equations that describe the power system [6].

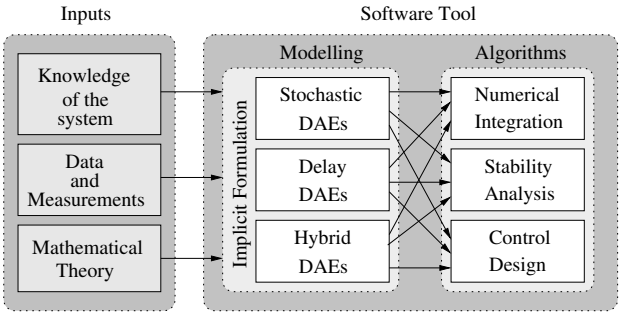


Fig. 1: Architecture of the proposed software tool for power system analysis.

The knowledge of the behaviour of power systems, real-world data, measurements and mathematical theory form the conceptual and practical bases of this paper (see Fig. 1). These are translated into efficient software tools through which all modelling, stability analysis, and control design are implemented and tested. Figure 1 also shows the three strands of the developed software tool: (i) Stochastic Differential-Algebraic Equations (SDAEs); (ii) Delay Differential-Algebraic Equations (DDAEs); and Hybrid Differential-Algebraic Equations (HDAEs), respectively.

This paper aims at defining a methodological paradigm for the modelling, stability analysis and control of power systems and smart grids. The paper also discusses the formalism to describe

the diversity of phenomena and events that compose an electrical power system. The result is the proposal of a power system model based on stochastic, functional, hybrid implicit differential-algebraic equations.

1.2 Literature Review

This section describes the state of the art, at the time of the beginning of the initial development of the software tool, of each strand, namely formalism for stochastic, functional and hybrid differential algebraic equations.

1.2.1 Stochastic Differential-Algebraic Equations (SDAEs): Power system variables evolve in different time scales. To take into account this behaviour, power systems are traditionally modelled as a set of Differential-Algebraic Equations (DAEs). Due to the stiffness of this model, implicit numerical methods should be used in simulations to avoid numerical instability. On the other hand, if Stochastic Differential Equations (SDEs) are used to model random perturbations in power systems, the system model becomes a set of SDAE. Therefore, solving SDAE models involves dealing with both stochastic terms and stiffness. Observe that the numerical integration of SDAEs is much more complex and computationally demanding than in the case of SDEs. With this regard, in [7], the adequacy of different implicit fixed step size numerical methods for SDAEs is discussed. In the context of electronic circuit simulation, [8] shows that implicit numerical methods with fixed step size used to solve SDEs are also suitable for being applied to SDAEs.

For the reasons outlined above, the literature on power system analysis is mostly limited to SDEs. Traditionally, the focus has been on modelling load behaviours [9–11]. In [12] SDEs are used as a planning tool for power systems. In particular, SDEs are used to model small perturbations in both system loads and transmission line parameters. A similar approach is used in [13] and [14] to analyse power system dynamics, where discrete perturbations are considered in switching events due to the operation of tap-changing transformers. The effect of stochastic processes on power system voltage stability is studied in [15–17]. In [18], both load and wind power production are modelled with SDEs to address the problem of power system supply-demand balance in an hourly time frame. More recently, in [19] random loads are modelled through SDEs which are included directly in the algebraic equations of a power system model. The problems related to the appearance of singularities in the model resulting from this approach are investigated in [20]. Finally, stochastic transient stability is discussed in [4], and the application of SDEs to wind speed modelling is analysed in [21]. In the same context, in [22] an implicit variable step size scheme for SDAEs is proposed. More recently, an implicit fixed step size method for SDAEs with stiffness in both the deterministic and the stochastic parts of the model has been proposed in [23].

1.2.2 Delay Differential-Algebraic Equations (DDAEs): Time delays arise in a wide variety of physical systems and their effects on stability have been carefully investigated in several engineering applications, such as signal processing and circuit design [24, 25]. Nevertheless, thus far, research has been lacking on the effects of time delays on power system stability. As a matter of fact, time delays are generally ignored. An exception to this rule is [26], which presents a model of long transmission lines in terms of DDAEs.

In recent years, wide measurement areas and the recent application of Phasor Measurement Unit (PMU) devices make remote measurements necessary, which has led to some research on the effects of measurement delays. For example, [27] and [28] present a robust control scheme, considering the effect of time delays, for wide-area Power System Stabilisers (PSSs), and [29] tackles the issue of time domain integration of DDAEs. The effects on small signal stability of delays due to PMU measurements are studied in [30], based on a probabilistic approach.

Existing studies on stability and control of delayed power system equations can be divided into two main categories: (i) time-domain methods, and (ii) frequency-domain methods.

Time-domain methods: These methods are based on Lyapunov-Krasovskii stability theorem and Razumikhin theorem. The application of time-domain methods allows for the definition of robust controllers (e.g., H_∞ control) and the capability to deal with uncertainties and time-varying delays. However, the conditions of the Lyapunov-Krasovskii stability theorem and Razumikhin theorem are only sufficient and cannot be used to find the delay stability margin. Moreover, it is necessary to find a Lyapunov functional or, according to Razumikhin theorem, a Lyapunov function that bounds the Lyapunov functional. Hence, in the nonlinear case, the applicability of time-domain methods strongly depends on the ability to define a Lyapunov function (i.e., the same limitation as in the case of DAE systems). The application to power system analysis are limited to small linearised test systems [31, 32]. If the DDAE is linear or is linearised about an equilibrium point, finding the Lyapunov function, in turn, implies finding the solution of a Linear Matrix Inequality (LMI) problem [33, 34]. A drawback of this approach is that the size and computational burden associated with LMIs drastically increases with the size of the DDAEs. As a matter of fact, LMI-based analysis has become computationally tractable only in the last two decades [34]. However, despite their large computational burden, in recent years, LMI-based approaches have been applied to several practical problems (see, for example, [6]).

Frequency-domain methods: These methods mainly involve the evaluation of the roots of the characteristic equation of the retarded system [30, 35, 36]. This approach is in principle exact but due to the difficulty in determining the roots of the characteristic equation, the analysis is limited to one-machine infinite-bus systems. Although an exact explicit analytic method based on the Lambert W function can be applied to simple cases [37], the analytic solution of the eigenvalue problem of delayed systems cannot be found for practical power systems. Thus, frequency-domain methods rely on approximated numerical techniques. A possible approach is based on the discretisation of the solution operator of the characteristic equation [38]. Other methods estimate the infinitesimal generator of the solution operator semi-group [39], and the solution operator approach via linear multi-step time integration of retarded systems without any distributed delay term [40]. Yet other approaches apply a discretisation scheme based on Chebyshev nodes [41, 42]. These methods are based on a discretisation of the Partial Differential Equation (PDE) representation of the DDAEs. The implementation of such discretisation is surprisingly simple while results proved to be accurate. The idea is to transform the original DDAE problem into an equivalent PDE system of infinite dimensions. Then, instead of computing the roots of retarded functional differential equations, one has to solve a finite, though possibly large, matrix eigenvalue problem of the discretised PDE system.

1.2.3 Hybrid Differential-Algebraic Equations (HDAEs): An area that appears of particular relevance for power systems that heavily rely on telecommunications is the study of the effect of discontinuities (e.g., breaker operations, tap changer positions and power electronics switching) on power system behaviour. Pioneering work in this field was provided by Hiskens, which outlines a formalisation of grazing bifurcations [43, 44].

The study of HDAEs, i.e., dynamic equations with both continuous and discrete variables, can be roughly divided in to two categories: (i) equations with *discontinuous right-hand side*, where the discrete variables are due to structural changes, such as the hard limits of the controllers; and (ii) *behavioural models*, i.e., equations where the discrete variables approximate a complex model whose details and dynamics are not relevant for capturing the overall system dynamic, i.e., the modelling of mosfets as simple switches.

Differential equations with discontinuous right-hand side: The analysis of differential equations with discontinuous right-hand sides has been a subject of intense research, and well-established techniques were developed in this area in the second half of the past century (see, for example, [45–47]). In particular, some fundamental work in this area was the formalism introduced by Filippov in [46]

for dynamic systems with switching manifolds. Filippov equations are a powerful tool to define discontinuities as well as the imperfect behaviour of measured control signals (e.g., gaps and loss of information). Attempts to introduce the rigorous formalism by Filippov into power systems have not been conducted thus far. The implementation of a software tool with this capability will allow for a better understanding of the behaviour of the impact of discontinuities on power system operation.

Behavioural models: The well-assessed formalism based on Discrete-Events systems (DEVS) and the resultant extensions to hybrid continuous and discrete-event systems has formed the source of an extremely vast and diverse literature [48, 49]. A similar approach, based on *behavioural variables* is that used in electronic circuit analysis and hardware description languages (HDL), e.g., Verilog HDL [50] and, notably, Verilog-A that mixes analog and digital models [51]. Discrete-events and/or behavioural models are a convenient and efficient way to describe digital systems. Recently, experts in DEVS have been studying power systems and their implementations using the DEVS formalism [52, 53]. A relevant example is the software tools implemented at the Oak Ridge National Laboratory, USA, as well as references in [54].

1.3 Contributions

This work proposes the formulation of power systems as a set of Stochastic Functional Hybrid Implicit Differential-Algebraic Equations (SFHI-DAEs) and describes a variety of modelling and efficient simulation techniques to integrate, study the stability and design robust controllers for this kind of dynamic models. The main contributions of this paper are the following.

- To provide a novel paradigm for the analysis and simulation of power systems. This goal is achieved by modelling power systems as a set of discrete-event, behavioural, stochastic, functional, discontinuous implicit differential-algebraic equations as opposed to the conventional classical model that is based on deterministic continuous differential-algebraic equations.
- To remove some conventional simplifications and hypotheses that were assumed in power system models several decades ago and then seldom discussed again. These are, for example, the effect of discontinuities, digital signals, stochastic processes, etc. Apart from technical complexity, there is also the challenge to make practitioners aware of common simplifications that are made and of the importance of re-evaluating commonly-accepted models.
- To combine concepts from applied mathematics (e.g., fractional calculus) and nonlinear control theory (e.g., utilization of delays to improve the stability of the system), as well as concepts from computer science and, in particular, modern modelling & simulation techniques to reformulate the architecture and the basis on which power system simulators are built.
- To discuss modelling limitation, implementation challenges and numerical issues of the proposed SFHI-DAE formulation, as well as a vision for future work in the field of power system modelling and simulation.

All contributions are supported with simulations based on a dynamic model of the All-Island Irish Transmission System (AIITS). These simulations are obtained with the software tool Dome [55].

1.4 Organisation

The remainder of this paper is organised as follows. Section 2 describes the proposed power system model as a set of SFHI-DAEs. Section 2 also provides relevant remarks on the proposed model, including methodological approaches, challenges encountered during the development of the software tool and open questions and unresolved issues for future implementation and research. With this aim, this section provides relevant references where the interested reader can find additional examples and case studies carried out by the authors that illustrate the techniques discussed in this paper. Section 3 presents a variety of case studies, based on the AIITS that illustrates the various features and capabilities of the proposed SFHI-DAE-based power system model. Finally, Section 4 summarises the main conclusions and the lesson learnt from the project AMPSAS and draws relevant recommendations for future work.

2 Modelling

The conventional power system model for transient stability analysis consists of a set of explicit DAEs [56–58]:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$ denote the state and algebraic variables, respectively; and $\boldsymbol{f}$ and $\boldsymbol{g}$ are non-linear differential and algebraic equations, respectively.

Equations (1) represent the model that, with various degrees of simplifications and with various techniques, has been utilised for more than a century for the transient stability analysis of power systems. This models is specifically designed to account for the time scales of the electromechanical dynamic response of synchronous machines and their primary controllers while neglecting electromagnetic transients.

Model (1) is often referred to as Root-Mean Square (RMS) or Quasi-Steady State (QSS) model because voltages and currents are assumed to be *slowly varying* phasors. This is one of the main idiosyncrasies of (1), namely the fact that the frequency is assumed to be constant in the definition of admittances and impedances, but phasors have variable (algebraic) phases and synchronous machines rotor angles and speeds are state variables. This apparent inconsistency often confuses researchers coming from circuit analysis and control theory but is, in effect, a very sensible approximation for conventional power systems whose dynamics are dominated by the synchronous machines and their primary controllers [59].

2.1 Intepretation of Algebraic Variables and Equations

2.1.1 Singular Perturbation Approach: There are various ways to interpret the algebraic equations and variables in (1). Probably the most intuitive one is to consider $\boldsymbol{y}$ as *demoted* state variables, i.e., states with a infinitely fast response. This might be often the case, especially when thinking of the voltages and current in transmission lines as these, in principle, are the states of the shunt capacitive charging and series inductive elements, respectively, of the lines. This interpretation justifies the *singular perturbation approach* [10, 60]:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \\ \boldsymbol{\epsilon} \odot \dot{\boldsymbol{y}} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \end{aligned} \tag{2}$$

where $\boldsymbol{\epsilon} \in \mathbb{R}^m$ is a vector of *small* positive numbers and $\odot$ is the element-by-element product. This interpretation, however, does not take into account "auxiliary" variables, i.e., variables and equations that are introduced in the model only to simplify the formulation, for example, in rectangular coordinates, it is often needed to define the magnitude of the voltage, hence the equation:

$$0 = v_r^2 - v_i^2 - v^2\,, \tag{3}$$

where v_r and v_i are the real and imaginary components of the QSS voltage phasor and v its magnitude. Of course, one can invent a dynamic for this equations by defining:

$$\epsilon\, \dot{v} = v_r^2 - v_i^2 - v^2\,, \tag{4}$$

but this dynamic is not physical and constitutes, in effect, an arbitrary modification of the overall system dynamic behaviour. This might

not be an issue if ϵ is sufficiently small but using very small values for ϵ also contributes to the *stiffness* (i.e., the spread of the time scales of the dynamics of the system) of the differential equations and, in turn, can make more challenging their numerical integration.

2.1.2 Algebraic Equations as Constraints: Another, often useful interpretation of $\boldsymbol{g}$ is as *constraints* of the differential equations. According to this interpretation, the purpose of algebraic variables $\boldsymbol{y}$, thus, is that of reducing the domain of $\boldsymbol{x}$ to a subset $\mathcal{X} \in \mathbb{R}^n$ which is not known a priori but depends on the equations $\boldsymbol{g}$. This interpretation has its natural ally in the implicit function, which suggests that – at least in theory – the algebraic equations can be expressed as:

$$\boldsymbol{y} = \boldsymbol{g}^{-1}(\boldsymbol{x}, t)\,, \tag{5}$$

which allows rewriting (1) as a set of Ordinary Differential Equations (ODEs):

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{g}^{-1}(\boldsymbol{x}, t), t)\,. \tag{6}$$

In practice, as it has to be expeced, (5) can almost never be found explicitly. Interestingly, from the numerical point of view, being able to find $\boldsymbol{g}^{-1}$ is not necessary nor, in fact, desirable. This can be easily understood considering a linear system in the form:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{F}_x\,\boldsymbol{x} + \mathbf{F}_y\,\boldsymbol{y}\,, \\ \mathbf{0} &= \mathbf{G}_x\,\boldsymbol{x} + \mathbf{G}_y\,\boldsymbol{y}\,. \end{aligned} \tag{7}$$

This system can be obtained as an approximation of (1) or, more commonly, its linearisation at a stationary point. In the latter case, the variables are effectively the variations around the stationary point rather then the actual variable of (1). If $\mathbf{G}_y$ is not singular, $\boldsymbol{y}$ can be expressed as a function of $\boldsymbol{x}$ and (7) can be written as:

$$\dot{\boldsymbol{x}} = \mathbf{F}_x - \mathbf{F}_y\,\mathbf{G}_y^{-1}\,\mathbf{G}_x\,\boldsymbol{x} = \mathbf{A}_0\,\boldsymbol{x}\,, \tag{8}$$

While apparently the resulting system has smaller size then (7), from the numerical point of view, (8) is often more computationally demanding because $\mathbf{G}_y^{-1}$ and, hence, $\mathbf{A}_0$, tends to be dense, even if all original matrices are very sparse. This is the common situation in power systems. In turn, for what concerns numerical performance, the order of the system is much less important than the sparsity of the (Jacobian) matrices that describe the system itself. This appraisal is one of the rationales for the proposals of an implicit formulation in this paper. In the other hand, the discussion above does not imply that one should arbitrarily define auxiliary algebraic variables just for the sake of increasing sparsity. The optimal balance between the size of m and the sparsity degree of the Jacobian matrices of the system is still an open question and, for now, is often solved heuristically, e.g., by trial and error.

2.1.3 Singular Differential Equations: A third and more mathematical way to interpret (1) is as *singular differential equations*, i.e., differential equations for which some of the coefficients of the time derivatives are null. In the case of (1), the null coefficients are those that multiply the vector $\dot{\boldsymbol{y}}$, namely:

$$\begin{aligned} \mathbf{1} \odot \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \\ \mathbf{0} \odot \dot{\boldsymbol{y}} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \end{aligned} \tag{9}$$

This is the approach utilised in some monographs on differential equations and, very recently, on matrix pencils, which often do not distinguish between states and algebraic variables [61]. This interpretation promotes the utilisation of the set of DAEs in (1) as is, thus preserving the total order of $n + m$ of the system. This is the approach utilised in this paper.

One can argue that (1) is the stiffest system possible as it includes infinitely fast dynamics, but this is in fact not a major problem if the DAEs are integrated simultaneously rather then sequentially. Moreover, this interpretation allows assigning an *index* to the DAE. Simply stated, ODEs are index 0, whereas DAE are index 1 if the first time derivative of $\boldsymbol{g}$ leads to well defined ODEs, as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \\ \mathbf{0} &= \boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}, t)\,\dot{\boldsymbol{x}} + \boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, t)\,\dot{\boldsymbol{y}}\,, \end{aligned} \tag{10}$$

where $\boldsymbol{g}_{\boldsymbol{y}} = \nabla_{\boldsymbol{y}}^T \boldsymbol{g}$ and $\boldsymbol{g}_{\boldsymbol{x}} = \nabla_{\boldsymbol{x}}^T \boldsymbol{g}$. Equations (10) can be rewritten implicitly as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \\ \dot{\boldsymbol{y}} &= -\boldsymbol{g}_{\boldsymbol{y}}^{-1}(\boldsymbol{x}, \boldsymbol{y}, t)\,\boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}, t)\,\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \end{aligned} \tag{11}$$

which, of course, can be defined only if $\boldsymbol{g}_{\boldsymbol{y}}^{-1}$ is never singular. If $\boldsymbol{g}_{\boldsymbol{y}}^{-1}$, then one can derive the second of (10) again. The index is in turn the number of times the DAEs have to be derived to obtain a set of ODEs. It is important to note that the actual calculation of (11), even if possible, is not needed and is never efficient (for the reason discussed above on the dense nature of $\boldsymbol{g}_{\boldsymbol{y}}^{-1}$). The formal definition of 11, on the other hand, is crucial for the ability of numerical methods to integrate efficiently (1).

Interestingly, the non-singularity of $\boldsymbol{g}_{\boldsymbol{y}}$ always holds in practical computer implementations and is assumed in all developments discussed in this paper. The statement "$\boldsymbol{g}_{\boldsymbol{y}}$ is full rank in practice" is based on the experience of the authors, despite the relative large number of works that has focused on the discussion of the points at which $\boldsymbol{g}_{\boldsymbol{y}}$ is singular (e.g., [62] and [63]). In the numerical analysis of power systems, in fact, the singularity of $\boldsymbol{g}_{\boldsymbol{y}}$ is really never encountered *by accident* during a time domain simulation or at an equilibrium point. This is principally due to the representation of floating point numbers. Moreover, anyone that has ever implemented an Newton-Raphson algorithm knows that the golden rule whenever the factorisation of a matrix is involved, is to add to its diagonal a small number (10^{-24} works well for 64-bit floating point number qrepresentation) to avoid numerical issues. For this reason, in the remainder of this paper, $\boldsymbol{g}_{\boldsymbol{y}}$ is always assumed to be full rank.

2.1.4 Semi-implicit and Implict Formulations: Equations (10) can be rewritten as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \\ -\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, t)\,\dot{\boldsymbol{y}} &= \boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}, t)\,\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\,, \end{aligned} \tag{12}$$

or equivalently:

$$\begin{bmatrix} \boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, t) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t) \\ \boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}, t)\,\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t) \end{bmatrix}\,, \tag{13}$$

or, equivalently:

$$\mathbf{T}(\boldsymbol{z}, t)\,\dot{\boldsymbol{z}} = \boldsymbol{\psi}(\boldsymbol{z}, t)\,, \tag{14}$$

where $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$ and $\mathbf{T}$ and $\boldsymbol{\psi}$ are the left and right terms that appear in (13). Equations (14) which is a *semi-implicit* form of (1).

It is important to note that one can write any device model in a semi-implicit form and that this does not imply necessarily having to differentiate the algebraic equations $\boldsymbol{g}$. This is thoroughly discussed in [64], which also discusses how the semi-implicit formulation can be useful to increase the sparsity of the Jacobian matrices of the system without introducing any simplification.

It is also important to note that $\mathbf{T}$, which can be interpreted as a sort of generalised *mass matrix*, does not need to be full rank for (14) to be integrated using a simultaneous implicit integration scheme such as the Implicit Trapezoidal Method (ITM) or the Backward Differentiation Formulas (BDFs). In [64], in fact, the semi-implicit form (14) is obtained without calculating the Jacobian matrices of $\boldsymbol{g}$ and without the need to introduce $\dot{\boldsymbol{y}}$. Instead, $\dot{\boldsymbol{x}}$ is utilised to simplify the expressions of $\boldsymbol{f}$ and $\boldsymbol{g}$.

The concept above is better illustrated through an example. Let us consider the double lead-lag shown in Fig. 2. A conventional explicit

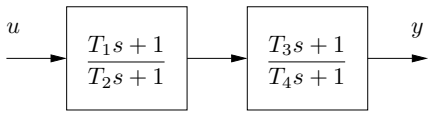


Fig. 2: Series of two lead-lag transfer functions.

formulation of the DAEs that describe this double lead-lag is:

$$\begin{aligned} \dot{x}_1 &= (u - x_1)/T_2 \,, \\ \dot{x}_2 &= [T_1(u - x_1)/T_2 + x_1 - x_2]/T_4 \,, \\ 0 &= T_3[T_1(u - x_1)/T_2 + x_1 - x_2]/T_4 + x_2 - y \,. \end{aligned} \tag{15}$$

whereas the semi-implicit form can be written as:

$$\begin{aligned} T_2\,\dot{x}_1 &= u - x_1 \,, \\ T_4\,\dot{x}_2 - T_1\,\dot{x}_1 &= x_1 - x_2 \,, \\ -T_3\,\dot{x}_2 &= x_2 - y \,. \end{aligned} \tag{16}$$

Equations (16) are, overall, "simpler" than (15) as the input u does not *propagates* through all equations and there are less terms. Moreover, no division by the time constant is needed in (16), which allows *downgrading* state variables to algebraic ones simply by setting to zero the time constnats for which they are multiplied. Finally, the Jacobian matrices of (16) are also sparser than those of (15) at the price, however, of a non-diagonal and non-symmetrical matrix $\mathbf{T}$.

The property of the implicit formulation to increase the sparsity of the equations and their Jacobian matrices is particularly useful when one utilises techniques that involve series of several lead-lags such as the typical approximations utilised to represent fractional order derivatives [65] (see Section 2.3) or the Padé approximants of delayed variables [66]. In the specific case of lead-lag series, for example, the implicit formulation leads to a tridiagonal Jacobian matrix (number of non-zeros elements $3n - 2$, where n is the order of the matrix), whereas the explicit formulation leads to a triangular one (number of nonzero elements $n(n + 1)/2$).

In general, it is not even necessary that the time derivatives are separated from the right hand side, which leads to the *implicit form*:

$$\mathbf{0} = \boldsymbol{\varphi}(\boldsymbol{z}, \dot{\boldsymbol{z}}, t) \,. \tag{17}$$

It is relevant to complete this discussion by observing that the differentiation of (17) at a stationary point leads to:

$$\mathbf{0} = \boldsymbol{\varphi}_{\boldsymbol{z}} \Delta \boldsymbol{z} + \boldsymbol{\varphi}_{\dot{\boldsymbol{z}}} \Delta \dot{\boldsymbol{z}} = \mathbf{A}\,\boldsymbol{z} + \mathbf{E}\,\dot{\boldsymbol{z}} \,, \tag{18}$$

which leads to the *matrix pencil*:

$$\mathbf{A} + \mathbf{E}\,s \,. \tag{19}$$

where s is the variable of the Laplace transform. A relevant applications of the pencil (19) is the small-signal stability analysis of the linear(ised) set of implicit DAEs, which is the solution of the generalised eigenvalue problems:

$$\det(\mathbf{A} + \mathbf{E}\,\lambda) = 0 \,, \tag{20}$$

which offers some benefit (sparser matrices) and some drawbacks (very few available solvers and libraries able to treat the general problem with $\mathbf{A}$ and $\mathbf{E}$ both asymmetric [67]). The interested reader can find a comprehensive discussion on the utilisation of matrix pencils and eigenvalue problems, including dual, nonlinear and singular ones, for power system small-signal stability analysis in the monograph [68].

The implicit form (17) is the formulation utilised in the proposed power system model and in all cases studies. We further elaborate on this form in Sections 2.4.

2.2 Specialised Models

Before introducing the proposed model, we discuss some relevant *specialised models*, namely models that have been proposed in the literature and focus on a specific modelling detail of the more general model proposed in this work. Every model presented in the remainder of this section, thus, can be viewed as a special case of the model given in Section 2.4.

2.2.1 Frequency Dependent Model: An ongoing debate in the power system community is the discussion whether the utilisation of model (1) is adequate at all given the numerous drastic changes in the dynamics and controls that have been introduced in the last couple of decades in power systems [1, 2]. Power system converters are much faster than synchronous machines and, while do not provide inertia, they are also exempt from technical constraints such as the relatively slow dynamics of the governors of conventional power plants.

Among the various proposed approaches are multi-scale models [69, 70], dynamic phasors [71, 72], and EMT models with integration methods optimised for the simulation of large power systems [73]. Here, we do not engage in this debate, nor judge the merit of each approach. In the experience matured by the authors, however, we have found that high order harmonics are rarely any issue for the overall dynamic behaviour of the system and, in any case, these harmonics can be studied separately. This observation applies to large interconnected power systems, not to microgrids, whose dynamics behaviour constitute a world apart and should not be mixed up with that of transmission systems.

In this work, we consider exclusively the frequency dependence of the various components of the system and a selection of *fast* dynamics through the concepts of *frequency divider* [74] and *Park vectors* [75].

The frequency divider is a continuum-approximation-based technique to estimate the frequency variations at the buses of the grid based on synchronous machines rotor speeds. This approach has been utilised in [76] and some results are briefly discussed in Section 3.1. The main limitation of the original definition of the frequency divider is the assumption that only synchronous machines are able to modify/impose the frequency at their point of connection. This has been demonstrated not to be the case in more recent works by the same authors [77, 78]. This limitation can be solved, case by case, by patching the frequency divider formula for devices other than synchronous machines. This has been done, for example, in [79].

The Park vector, on the other hand can be viewed as the simplest approximation of dynamic phasors, in that it retains only the fundamental frequency. However, it also retains the fast dynamics of transmission lines and of the magnetic fluxes of rotating machines. Park vector are compatible with the average models of power electronic converters [80] and makes possible to study the dynamic interaction of the DC and AC sides of the converters [81]. In [82], the two concepts of frequency divider and Park vectors are combined to allow a precise, analytical definition of the frequency variations at grid buses. This, however, comes at the price of introducing a new quantity, namely the *complex frequency* that can be involved to calculate.

2.2.2 Stochastic Differential-Algebraic Equations: Stochastic differential equations have been utilised in power systems since the 1908s, at least in academic works, but with the strong limitation of including perturbations only in the differential equations. This is limitation is a consequence of the fact that it is not known how to include stochastic processes in algebraic equations or, which is the same, to singular differential equations [83, 84]. Early attempts to include noise in the algebraic variables have utilised the singular perturbation approach which, as said above, consists in assigning a fictitious dynamic to algebraic variables [10, 85].

More recently, the first author has proposed a general model in [86] that is able to include additive noise to both state and algebraic variables. Since parameters can always be made algebraic variables with a dummy equation, say $0 = \alpha - y$, also parameters can be thus made vary stochastically. The model proposed in [86] is the

following:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \dot{\boldsymbol{\eta}}) \,, \\ \mathbf{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}) \,, \\ d\boldsymbol{\eta} &= \boldsymbol{a}(\boldsymbol{\eta}, t)dt + \boldsymbol{b}(\boldsymbol{\eta}, t) \odot d\boldsymbol{w}(t) \end{aligned} \tag{21}$$

$\boldsymbol{\eta} \in \mathbb{R}^q$ are the states of the stochastic processes; $d\boldsymbol{w} \in \mathbb{R}^r$ are the Wiener process increments; $\boldsymbol{a}$ and $\boldsymbol{b}$ are the drift and the diffusion terms of the Wiener processes, respectively.

The stochastic processes $\boldsymbol{\eta}$ are expressed in terms of variational equations rather then differential equations that describe deterministic dynamic models. This is because stochastic processes are conceptually different from conventional ODEs and also require a specific integration scheme. In [86], the Euler-Maruyama method is utilised, which is the simplest first order integration scheme but is equivalent to higher order schemes, e.g., the Milton one, if the diffusion term is constant. A positive aspect of the integration of SDEs is that the diffusion term is integrated separately and is delayed by one step with respect to the drift. This means that the integration scheme of the deterministic part of (21), namely the first two equations and $\boldsymbol{a}$, can be different from the one utilised for the diffusion $\boldsymbol{b}$. This delay, however does not impact on the accuracy as, in general, $\boldsymbol{a}$ and $\boldsymbol{b}$ do not depend on $\boldsymbol{x}$ and $\boldsymbol{y}$.

It is worth noticing that, in (21), only first time derivatives of the stochastic processes appears in the differential equations $\boldsymbol{f}$ [86]. The rationale for this is readily explained. If noise is additive, then

$$\boldsymbol{x} = \tilde{\boldsymbol{x}} + \mathbf{U}\,\boldsymbol{\eta} \,, \tag{22}$$

where $\tilde{\boldsymbol{x}}$ is the deterministic part of the vector of randomly perturbed state variables $\boldsymbol{x}$ and $\mathbf{U}$ is a $n \times q$ matrix that defines the weights of the stochastic processes onto the differential equations. The differentiation of (22) yields:

$$\begin{aligned} d\boldsymbol{x} &= \tilde{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{y}, t)dt + \mathbf{U}\, d\boldsymbol{\eta} \\ &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \dot{\boldsymbol{\eta}}, t)dt \,, \end{aligned} \tag{23}$$

where $\tilde{\boldsymbol{f}}$ is the deterministic part of the differential equations $\boldsymbol{f}$. Equation (23) is expressed in an explicit form but can be conveniently rewritten in a semi-implicit form:

$$d\boldsymbol{x} - \mathbf{U}\, d\boldsymbol{\eta} = \tilde{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{y}, t)dt \,, \tag{24}$$

which is computationally more convenient than using the expressions of $d\boldsymbol{\eta}$ into $\boldsymbol{f}$.

The properties of stochastic processes that the SDEs that define variables η have to reproduce are probability distribution, autocorrelation and correlation with other processes. The probability distribution also implies the mean, the standard deviation and higher order momenta of the process. Since the Wiener process has zero expectation, the mean of $\boldsymbol{\eta}$, say $\langle\boldsymbol{\eta}\rangle$, is the vector that satisfies the condition $\boldsymbol{a}(\langle\boldsymbol{\eta}\rangle) = \mathbf{0}$. Imposing the standard deviation is more involved as it is intertwined with the probability distribution and the autocorrelation. References [87–89] discuss various methods to impose these properties considering arbitrary probability densities and/or autocorrelation functions and the reader is referred to these works for details on the implementation of these techniques. A common feature of all techniques is to have bounded standard deviation in stationary conditions. This property is observed in measurement data and is generally imposed through a *mean-reverting* process, such as the Ornstein-Uhlenbeck process [86].

While mean, standard deviation and probability distribution are relatively well-understood concepts, more involved (and much less commonly found in works on SDEs) is a discussion on the effects of the autocorrelation. This can be interpreted as an equivalent for stochastic processes of the time constant of a deterministic dynamics. The autocorrelation, in turn, defines how fast or slow a process can vary from one time lag to another and, in turn defines the harmonic content of the process itself. For this reason, two processes with identical probability distribution but different autocorrelations may have significantly different effects on the dynamic response of a system [90].

The last property, namely correlation among processes is yet another aspect of SDEs that has not been fully discussed in the literature on power system dynamic performance. To account for correlation, one has first to determine the correlation matrix, say $\mathbf{R} \in \mathbb{R}^{r\times r}$ of the r stochastic processes. $\mathbf{R}$ is symmetric and has 1's on its diagonal, as follows:

$$\mathbf{R} = \begin{bmatrix} 1 & R_{1,2} & R_{1,3} & \dots & R_{1,n} \\ R_{2,1} & 1 & R_{2,3} & \dots & R_{2,n} \\ R_{3,1} & R_{3,2} & 1 & \dots & R_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{n,1} & R_{n,2} & R_{n,3} & \cdots & 1 \end{bmatrix} . \tag{25}$$

Note that if the processes are fully uncorrelated, then $\mathbf{R}$ is the identity matrix. The correlation matrix must then be decomposed, through Cholesky decomposition, into a matrix $\mathbf{C}$ that satisfies the following equation [91]:

$$\mathbf{R} = \mathbf{C}\,\mathbf{C}^T \,, \tag{26}$$

and, the correlated processes are computed as

$$d\boldsymbol{\zeta} = \mathbf{C}\, d\boldsymbol{w}(t) \,. \tag{27}$$

Finally, the (21) are modified by substituting $d\boldsymbol{w}$ with $d\boldsymbol{\zeta}$:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \dot{\boldsymbol{\eta}}) \,, \\ \mathbf{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}) \,, \\ d\boldsymbol{\eta} &= \boldsymbol{a}(\boldsymbol{\eta}, t)dt + \boldsymbol{b}(\boldsymbol{\eta}, t) \odot d\boldsymbol{\zeta} \,. \end{aligned} \tag{28}$$

The effect of correlation is that of increasing the probability that processes have similar trajectories. That is, if two processes are strongly correlated the probability that they increase (or decrease) at the same time is high. This means that correlated processes are more likely prone to create stability issues than two fully uncorrelated processes with same probability distribution.

The stochastic processes discussed so far are generated based on Wiener processes and are, thus, continuous. There are, however, some random events that are better modelled as jumps, such as load consumption variations [13, 92], the effect of tap-changer-under-load transformers [14, 93], and the effect of clouds on the active power generation of PV panels [94]. Random jumps can be also correlated as their continuous counterparts [95]. In mathematical terms, stochastic jumps have a similar expression as continuous stochastic processes and the two processes can be combined together [96]:

$$\begin{aligned} d\boldsymbol{\eta} &= \boldsymbol{a}(\boldsymbol{\eta}, t)dt + \boldsymbol{b}(\boldsymbol{\eta}, t) \odot [\mathbf{C}\, d\boldsymbol{w}(t)] \\ &\quad + \boldsymbol{c}(\boldsymbol{\eta}, t) \odot [\mathbf{K}\, d\boldsymbol{\jmath}(t)] \\ &= \boldsymbol{a}(\boldsymbol{\eta}, t)dt + \boldsymbol{b}(\boldsymbol{\eta}, t) \odot d\boldsymbol{\zeta} + \boldsymbol{c}(\boldsymbol{\eta}, t) \odot d\boldsymbol{\kappa} \,, \end{aligned} \tag{29}$$

where $d\boldsymbol{\jmath} \in \mathbb{R}^s$ are the jump increments; $\mathbf{K}$ is a $q \times s$ matrix that defines the correlation among the jumps and is obtained using same procedure as matrix $\mathbf{C}$; $d\boldsymbol{\kappa}$ are the correlated jump increments; and $\boldsymbol{c}$ is the diffusion terms of the Poisson stochastic processes.

From the implementation point of view, jumps are impulsive events that, when integrated lead to a staircase function. These events are slightly more involved to implement than Wiener processes. The terms $d\boldsymbol{\jmath}$ in (29) can be obtained through the combination of a Poisson process that determines how many events happen in a given period; a uniform distribution that determines the times at which the events happen in the given period; and a random process (e.g., Gaussian) that imposes the probability distribution of the amplitude of the events. Note that the parameter λ that defines the Poisson distribution – which is both the expected value and the standard deviation of the Poisson process – functions as the autocorrelation coefficient for the continuous stochastic processes.

2.2.3 Delay Differential-Algebraic Equations: As discussed in Section 1, another modification to the conventional model (1) that has been considered in the literature is the inclusion of delays. These are typically included in the system as index-1 Hessenberg forms, namely, the algebraic equations do not depend on delayed algebraic variables [29, 97]:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d, t)\,, \\ \mathbf{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, t)\,, \end{aligned} \tag{30}$$

where $\boldsymbol{x}_d = \boldsymbol{x}(t-\tau)$ and $\boldsymbol{y}_d = \boldsymbol{y}(t-\tau)$, with $\tau > 0$ are the delayed variables. The notation of (30) is for one delay, but it can be easily extended to multiples delays, as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d^1, \ldots, \boldsymbol{x}_d^\mu, \boldsymbol{y}_d^1, \ldots, \boldsymbol{y}_d^\mu, t)\,, \\ \mathbf{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d^1, \ldots, \boldsymbol{x}_d^\mu, t)\,, \end{aligned} \tag{31}$$

where $\tau^1, \ldots, \tau^\mu$ are the μ delays of the system. To simplify the notation, in the remainder of this paper, the vectors of state and algebraic variables with multiple delays are indicated as $\boldsymbol{x_d}$ and $\boldsymbol{y_d}$.

In the context of the project AMPSAS and in the general model proposed in Section 2.4, we extend the analysis to non-index-1 Hessenberg DDAEs, which are of the form:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d)\,, \\ \mathbf{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d)\,. \end{aligned} \tag{32}$$

In [98], the algebraic equations in (32) are viewed as constraints and this complicates significantly the small-signal stability analysis as throughly discussed in [98]. To explain this point, let us consider a stationary solution $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ of (32). This satisfies the conditions:

$$\begin{aligned} \mathbf{0} &= \boldsymbol{f}(\boldsymbol{x}_o, \boldsymbol{y}_o, \boldsymbol{x}_o, \boldsymbol{y}_o)\,, \\ \mathbf{0} &= \boldsymbol{g}(\boldsymbol{x}_o, \boldsymbol{y}_o, \boldsymbol{x}_o, \boldsymbol{y}_o)\,, \end{aligned} \tag{33}$$

where it has been used the fact that, in steady-state, $\boldsymbol{x}_{d,o} = \boldsymbol{x}_o$ and $\boldsymbol{y}_{d,o} = \boldsymbol{y}_o$. The differentiation of (30) at $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ yields:

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{f}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{y}}\Delta\boldsymbol{y} + \boldsymbol{f}_{\boldsymbol{x}_d}\Delta\boldsymbol{x}_d + \boldsymbol{f}_{\boldsymbol{y}_d}\Delta\boldsymbol{y}_d \tag{34}$$

$$\mathbf{0} = \boldsymbol{g}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \boldsymbol{g}_{\boldsymbol{y}}\Delta\boldsymbol{y} + \boldsymbol{g}_{\boldsymbol{x}_d}\Delta\boldsymbol{x}_d + \boldsymbol{g}_{\boldsymbol{y}_d}\Delta\boldsymbol{y}_d \tag{35}$$

where $\boldsymbol{g_y}$ is assumed to be full rank. Reference [98] shows that the characteristic equation of (32) linearised at the stationary point is given by:

$$\Delta\dot{\boldsymbol{x}} = \mathbf{A}_0\,\Delta\boldsymbol{x} + \mathbf{A}_1\,\Delta\boldsymbol{x}_d + \sum_{k=2}^{\infty}[\mathbf{A}_k\,\Delta\boldsymbol{x}(t-k\tau)]\,, \tag{36}$$

where

$$\begin{aligned} \mathbf{A}_0 &= \boldsymbol{f}_{\boldsymbol{x}} - \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}}\,, \\ \mathbf{A}_1 &= \boldsymbol{f}_{\boldsymbol{x}_d} - \boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}} + \boldsymbol{f}_{\boldsymbol{y}}\,\mathbf{N}\,, \\ \mathbf{A}_k &= [\boldsymbol{f}_{\boldsymbol{y}}\,\mathbf{M} + \boldsymbol{f}_{\boldsymbol{y}_d}]\,\mathbf{M}^{k-2}\,\mathbf{N}, \quad k \geq 2\,, \end{aligned}$$

with

$$\begin{aligned} \mathbf{M} &= -\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{y}_d}\,, \\ \mathbf{N} &= -\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}_d} - \mathbf{M}\,\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}}\,. \end{aligned}$$

The first matrix $\mathbf{A}_0$ is the well-known state matrix that is computed for standard models in the form of (1). The other matrices are not null only if the system include delays. The matrix $\mathbf{A}_1$ is found in any Delay Differential Equations (DDEs), while matrix $\mathbf{A}_2$ appears in (30) if both $\boldsymbol{f}_{\boldsymbol{y}_d}$ and $\boldsymbol{g}_{\boldsymbol{x}_d}$ are not null [97]. For index-1 Hessenberg forms, the term for $k > 2$ are null as $\boldsymbol{g}_{\boldsymbol{y}_d}$ and, hence, $\mathbf{M}$ are null. For non-index-1 Hessenberg forms, on the other hand, since $\boldsymbol{g}_{\boldsymbol{y}_d}$ is not null in general, there are infinitely many non-null matrices $\mathbf{A}_k$. Each delay τ generates thus a characteristic equation with infinite delays $k\tau, k = 1, 2, \ldots, \infty$. This clearly further complicates the numerical calculations of the roots of the characteristic equation of (36). It is relevant to note, in fact, that this characteristic equation has infinite solutions even for single-delay index-1 Hessenberg form. The characteristic equation of single-delay non-index-1 Hessenberg forms, on the other hand, has infinite infinite delays, each of which leading to infinite solutions. Finally, it is important to note that (36) can be studied only if the series converges, which happens if and only if, as demonstrated in [98], the spectral radius of $\mathbf{M}$ is strictly lower than 1.

The power system model proposed in this paper provides a more elegant way to study non-index-1 Hessenberg form DDAEs by considering an implicit formulation. This leads, after linearisation, to a generalised eigenvalue problem for which the calculation of the inverse of $\boldsymbol{g_y}$ is not required. Let us define $\boldsymbol{z}_d = (\boldsymbol{x}_d, \boldsymbol{y}_d)$ as the delayed generalised state vector $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$. Using an implicit formulation and considering for simplicity a single delay, the delayed version of (17) becomes:

$$\mathbf{0} = \boldsymbol{\varphi}(\boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{z}_d, t)\,, \tag{37}$$

and linearising at a stationary point $\boldsymbol{z}_o$ that satisfies $\dot{\boldsymbol{z}}_o = \mathbf{0}$ and

$$\mathbf{0} = \boldsymbol{\varphi}(\boldsymbol{z}_o, \mathbf{0}, \boldsymbol{z}_o)\,, \tag{38}$$

one obtains:

$$\mathbf{0} = \boldsymbol{\varphi}_{\boldsymbol{z}}\Delta\boldsymbol{z} + \boldsymbol{\varphi}_{\dot{\boldsymbol{z}}}\Delta\dot{\boldsymbol{z}} + \boldsymbol{\varphi}_{\boldsymbol{z}_d}\Delta\boldsymbol{z}_d\,, \tag{39}$$

which can be expressed in the form of (36), as follows:

$$\mathbf{E}\,\Delta\dot{\boldsymbol{z}} = \mathbf{A}_0\,\Delta\boldsymbol{z} + \mathbf{A}_1\,\Delta\boldsymbol{z}_d\,. \tag{40}$$

where $\mathbf{E} = -\boldsymbol{\varphi}_{\dot{\boldsymbol{z}}}$, $\mathbf{A}_0 = \boldsymbol{\varphi}_{\boldsymbol{z}}$ and $\mathbf{A}_1 = \boldsymbol{\varphi}_{\boldsymbol{z}_d}$. The implicit form, thus, does not lead to matrices $\mathbf{A}_k$ for $k \geq 2$ and does not introduce fictitious delays $k\tau$. Of course, this simplification is obtained at the cost of having a singular pencil ($\mathbf{E}$ is certainly singular) which complicates the numerical determination of the eigenvalues [68].

A special case of DDEs are Neutral Delay Differential Equations (NDDEs), i.e., DDEs that include the first time derivative of the delayed variables. Considering again for simplicity the single-delay case, one has:

$$\mathbf{0} = \boldsymbol{\psi}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{x}_d, \dot{\boldsymbol{x}}_d)\,, \tag{41}$$

which, using, a *descriptor model transformation* [99], can be equivalently rewritten as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{y}\,, \\ \mathbf{0} &= \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d)\,. \end{aligned} \tag{42}$$

Equations (42) are a set of non-index-1 Hessenberg form DDAEs. Thus, if one defines $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$, (42) can be rewritten in the same form as (37), whose linearisation leads to (40). Reference [100] discusses the small-signal stability of (41) using the transformation above with applications to circuits and systems.

While the small-signal stability analysis is significantly complicated by the presence of delays, numerical time domain integration methods can include delays in a relatively straightforward way. The only requirement is to store in some vector the past values of each delayed variable, for a time at least as big as the delay with which such a variable appears in the equations and then interpolate the value of the delay variables at the points evaluated by the integration scheme. It is worth noticing that, for implicit integration schemes that require the calculation of the Jacobian matrices of the system only the Jacobian matrices with respect to the current variables, namely $\boldsymbol{f_x}$, $\boldsymbol{f_y}$, $\boldsymbol{g_x}$ and $\boldsymbol{g_y}$, are required, not the Jacobian matrices with respect to delayed ones, thus resulting, in effect, in a sparser

matrix to factorise [29, 97]. This fact has been exploited in EMT analysis, for example, by utilising the inherent delays of long transmission lines to decouple the integration of circuits [101] or to take advantage of the different time scale of electromagnetic transients and controllers [102]. More recently, an application of the decoupling property of delays has been proposed during the development of the project AMPSAS for the transient stability analysis of power systems [103].

So far, we have considered exclusively constant delays. Communication systems however are characterised by variable, partially random delays [104]. Figure 3 shows a realisation of a realistic model of a Wide Area Measurement System (WAMS) delay, which is composed of a constant, a sawtooth and Gamma-distributed jumps. A communication delay is thus a relevant example of a physical process that combines stochastic, functional and discontinuous DAEs.

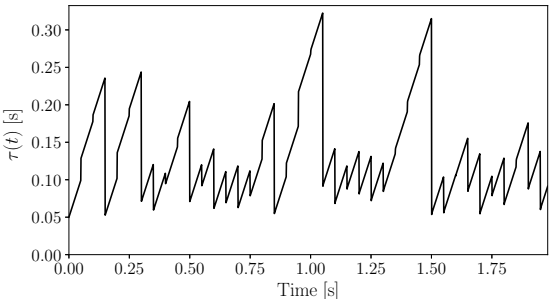


Fig. 3: Realistic model of a WAMS delay [104].

The implementation in a time integration tool of time-varying delays has almost same complexity as that of constant delays. This makes the numerical time-domain integration the best tool available for the analysis of the impact of time-varying time delays on power system dynamics. However, numerical integration schemes are known to be potentially able to modify the stability properties of the DDAEs [41]. It is, thus, desirable to have an alternative method to compare results.

During the development of the project AMPSAS, a considerable effort has been dedicated to the implementation of methods for the small-signal stability analysis of DDAEs with time-varying delays. This analysis is relevant because of the *quenching phenomenon* [105], namely the phenomenon for which "a system that is unstable with inclusion of a constant delay $\tau \in [\tau_{\min}, \tau_{\max}]$, can become stable for a time-varying delay $\tilde{\tau}(t)$ that varies in the same interval $[\tau_{\min}, \tau_{\max}]$, and *vice versa*" [106]. A breakthrough is provided by the work [107], that recognises that sufficiently fast varying delays are equivalent, from a stability point of view to distributed delays. Since distributed delays can be modelled as a series of infinitely many constant delays, this allows using available tools, e.g., PDE approximation followed by a Chebychev discretisation, for small-signal stability analysis of DDAEs [97, 108]. The computational burden of this analysis, however, increases as each time-varying delay yields infinitely many constants delays, each of which yields infinitely many roots of the characteristic equation of the linearised set of DDAEs.

Due to their negative effect on the dynamic performance of the system where they appear, delays have been largely studied in control theory. In the context of power system control, a conventional application is the design of controllers that are robust against the delays that appear in the system [6, 27] or techniques that compensate measurement delays [109]. As a final note on DDAEs, it is interesting to note that delays do not necessarily always create stability issues. An emerging application is the introduction – on purpose – of delays to improve the stability of the controllers. A pioneering work in this direction with applications to power systems is [110].

2.2.4 Hybrid Differential-Algebraic Equations: As discussed in the introduction, the ability to properly characterise discontinuities and behavioural models is an important aspect of power system modelling. These discontinuities and behavioural models lead to the inclusion of a vector of discrete variables in the set of DAEs. For each variable, an equation has to be added to the system but there is no unique nor well-accepted method to set-up these equations.

A relevant approach is the combination of a *hybrid automaton* and *Petri nets* proposed by Hiskens [111, 112]. The hybrid automaton is described by a finite set of discrete states; a collection of dynamical systems, one per each value of the discrete states; and a finite set of events. The Petri nets, in the other hand, are utilised to characterise unambiguously the discrete event activities. The approach described in [111] and several other papers by the same author is rigorous but requires to defines a large number of impulsive state variables, namely, variables that normally have $\dot{z} = 0$ and that change value, through a Dirac impulse, defined by a set of rules expressed in the form of conditional equations. Thus, this approach leads to singular Jacobian matrices (e.g., the row of the Jacobian matrix corresponding to equation $\dot{z} = 0$ is identically null all times), apart from increasing significantly the state space of the system itself.

During the execution of the project AMPSAS, we have adopted a different but equally rigorous approach based on the Filippov theory for differential equations with discontinuous right-hand side [46]. The starting point is common to the hybrid automaton approach, that is a switched dynamical system, which for generality, we assume to be in implicit form and function of the variables $\boldsymbol{z}$:

$$\mathbf{0} = \begin{cases} \boldsymbol{\varphi}_1(\boldsymbol{z}, \dot{\boldsymbol{z}})\,, & \text{when } h(\boldsymbol{z}, \dot{\boldsymbol{z}}) < 0\,, \\ \boldsymbol{\varphi}_2(\boldsymbol{z}, \dot{\boldsymbol{z}})\,, & \text{when } h(\boldsymbol{z}, \dot{\boldsymbol{z}}) > 0\,, \end{cases} \tag{43}$$

where h is an event function. Note the utilisation of the word *when* as opposed to the more common *if* in equation (43). This notation is inherited from the *when*-clause of the Modelica language and is substantially a time-driven *if*-clause [113]. The equations of the kind of (43) are particularly useful to define the status of the discrete variable an infinitesimal instant *before* and *after* a switch, often denoted as $\boldsymbol{h}^-$ and $\boldsymbol{h}^+$, respectively. This formulation, thus, facilitates the calculation of the *saltation matrix* which is required to properly obtain the fundamental matrix solution and trajectories sensitivities [114, 115].

Based on (43), the state space $\mathbb{R}^{n+m}$ can be split into two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ separated by a hyper-surface Σ where $\mathcal{R}_1$, $\mathcal{R}_2$ and Σ are characterised as:

$$\begin{aligned} \mathcal{R}_1 &= \{\boldsymbol{z} \in \mathbb{R}^{n+m} \mid h(\boldsymbol{z}) < 0\}\,, \\ \mathcal{R}_2 &= \{\boldsymbol{z} \in \mathbb{R}^{n+m} \mid h(\boldsymbol{z}) > 0\}\,, \\ \Sigma &= \{\boldsymbol{z} \in \mathbb{R}^{n+m} \mid h(\boldsymbol{z}) = 0\}\,, \end{aligned} \tag{44}$$

such that $\mathbb{R}^{n+m} = \mathcal{R}_1 \cup \Sigma \cup \mathcal{R}_2$, assuming that the gradient of h at $\boldsymbol{z} \in \Sigma$ never vanishes, $\boldsymbol{h}_{\boldsymbol{z}}(\boldsymbol{z}) \neq \mathbf{0}$ for all $\boldsymbol{z} \in \Sigma$.

The vector field on Σ is defined by Filippov continuation approach, known as *Filippov convex method* [46]. This method states that the vector field on the surface of discontinuity is a convex combination of the two vector fields in the different regions of the state-space:

$$\mathbf{0} = \begin{cases} \boldsymbol{\varphi}_1(\boldsymbol{z}, \dot{\boldsymbol{z}})\,, & \boldsymbol{z} \in \mathcal{R}_1\,, \\ \overline{co}\{\boldsymbol{\varphi}_1(\boldsymbol{z}, \dot{\boldsymbol{z}}), \boldsymbol{\varphi}_2(\boldsymbol{z}, \dot{\boldsymbol{z}})\}\,, & \boldsymbol{z} \in \Sigma\,, \\ \boldsymbol{\varphi}_2(\boldsymbol{z}, \dot{\boldsymbol{z}})\,, & \boldsymbol{z} \in \mathcal{R}_2\,, \end{cases} \tag{45}$$

where $\overline{co}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ is the minimal closed convex set containing $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$.

The added value of the Filippov theory is that it provides a systematic approach determine is what happens when the trajectory of $\dot{\boldsymbol{z}} = \boldsymbol{\varphi}_1(\boldsymbol{z})$, with $\boldsymbol{z}(0) = \boldsymbol{z}_o$ reaches at Σ in finite time. In turn, thus, the Filippov theory provides an alternative to the Petri nets utilised in [111] and other works by Hiskens.

The possibilities considered by Filippov are three: (i) transversal crossing, (ii) attractive sliding or repulsive sliding and (iii) smooth exit. Filippov formulated a first order theory to decide what to do in such kind of situation. The details of the general theory can be found in [46] and several illustrative examples that utilise circuits and power system models can be found in [116–119]. A particular advantage of the definition of the event functions $\boldsymbol{h}$ is that they can represent any nonlinear equation. This is particularly relevant when modelling control limiters with variable limits, such as the current limiters of the Voltage Sourced Converters (VSCs) utilised in in power-electronics based devices [120].

So far, we have discussed equations with discontinuous right-hand side, which are particularly adequate to model the hard limits of the controllers. Behavioural models (e.g., under-load tap-changer transformers) can also be formulated using Filippov theory (or hybrid automata and Petri nets) but this is a kind of overshooting. In the experience of the authors, it suffices to define simple *when*-clause rules that determine the values of the discrete values according to some event functions, as follows:

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_1\,, & \text{when } h(\boldsymbol{z}, \dot{\boldsymbol{z}}, t) < 0\,, \\ \boldsymbol{u}_2\,, & \text{when } h(\boldsymbol{z}, \dot{\boldsymbol{z}}, t) > 0\,, \end{cases} \tag{46}$$

where, for generality, h is assumed to be function not only of time and the current value of the states but also of the first time derivative. The resulting set of HDAEs able to take into account discontinuous right-hand side models as well as behavioural models is as follows:

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{\varphi}(\boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{u}, t)\,, \\ \boldsymbol{0} &= \boldsymbol{k}(\boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{u}, t)\,, \\ \boldsymbol{\varsigma} &= \boldsymbol{h}(\boldsymbol{z}, \dot{\boldsymbol{z}}, t)\,, \end{aligned} \tag{47}$$

where $\boldsymbol{u} \in \mathbb{R}^p$ are the discrete variables; $\boldsymbol{k}$ are the switching equations (46) that, in general, depend on all variables of the system and on time; and $\boldsymbol{h}$ are the event functions that decide the sign of the auxiliary variables $\boldsymbol{\varsigma} \in \mathbb{R}^{\xi}$. It is relevant to note that equations $\boldsymbol{k}$ are formally introduced here exclusively to account for the discrete variables $\boldsymbol{u}$. These equations are, in effect, algebraic equations, however, they differ from the equations $\boldsymbol{g}$ in that the Jacobian matrices of $\boldsymbol{k}$ do not form the matrix pencil of the linearised system and do not generate sensitivities, in the same way as no sensitivities with respect to the discrete variables $\boldsymbol{u}$ can be defined. In practice, equations $\boldsymbol{k}$ are implemented as *if*- or *when*-clauses.

The switching conditions of the discrete variables can be time- or state-driven. Time-driven events are straightforward to implement as it suffices to force the time domain simulation to evaluate a point right before and a point right after the event itself. That is, if an event is scheduled to happen at time t, the time domain simulation has to calculate a point at $t - \epsilon$ and another point at $t + \epsilon$. This is possible if the numerical integration scheme allows a variable time step. Otherwise some sort of interpolation is required. The interpolation is the typical solution for EMT simulations, which typically use a fixed time step. IN our experieince, however, variable time steps are more convenient for the transient stability analysis of power systems.

If the event is state-driven, then the identification of the exact time at which this event occurs is more involved. Functions $\boldsymbol{h}$ are evaluated during the whole time domain integration and so the sign of the elements of vector $\boldsymbol{\varsigma}$ are determined. Whenever a switching condition is identified, i.e., a change of sign of an element of $\boldsymbol{\varsigma}$ occurs, then proper actions are taken, i.e., the solver launches a Filippov theory-based procedure or switches a discrete variable.

With this regard, there are two relevant ways to proceed, either *time-stepping* or *event-driven* approaches [121]. These refer to the approach with which an event is treated during the iterations required to solve each step of the time domain integration (when using an implicit integration scheme). The time-stepping approach checks the signs of $\boldsymbol{\varsigma}$ at each iteration and switches the variables as soon as a change of sign is identified. The event-driven approach consists in completing the iterations and determining the point. Only then the event functions are evaluated and if any $\boldsymbol{\varsigma}$ has changed sign then the variable that is associated to the condition that shows the largest variations of the element of $\boldsymbol{\varsigma}$ is switched, the state variables are reset to the previous point and the calculation of the time step is repeated.

Both methods offer advantages and drawbacks. The time-stepping approach can be – slightly – more time consuming than the event-driven one if there are few events actually occurring during a simulation and the size of the event functions is large. On the other hand, the event-driven approach is more time consuming if many event occurs, especially if multiple events occur in the same time step. However, which method leads to the best performance of a given network and scenario can be hardly known *a priori*. In the experience matured in the execution of the project AMPSAS, the best solution is to have a software tool that can handle both approaches.

There is, however, a more crucial issue: the inclusion of discrete variables transforms the integration of DAEs from a deterministic problem into a combinatorial one. For example, if the system include p Boolean variables $\boldsymbol{u}$, then the system is characterised by 2^p possible combinations of the vector $\boldsymbol{u}$. It has to be expected that at least some of these combinations satisfy all the equations of the system – including constraints $\boldsymbol{k}$ – and are thus feasible solutions. In turn, there might be a variety of different feasible solutions of the system, even if all equations and rules that defines the discrete variables are deterministic. This conclusion does not arrive unexpected as, nonlinear equations are known to potentially have multiple solution and, in turn, discrete variables introduce a peculiar nonlinearity in the equations.

Another way is to interpret the stationary points of a set of HDAEs is as the optima of an optimisation problem. We note, in fact, that the stationary point of a set of DAEs can be viewed as the solution of an optimisation problem having as objective:

$$\begin{aligned} \text{min.}: \quad & \frac{1}{2}\dot{\boldsymbol{z}}^T\dot{\boldsymbol{z}}\,, \\ \text{s.t.}: \quad & \boldsymbol{0} = \boldsymbol{\varphi}(\boldsymbol{z}, \dot{\boldsymbol{z}}, t)\,. \end{aligned} \tag{48}$$

Then the determination of the equilibrium point of a set of HDAEs is equivalent to a mixed-integer optimisation problem, which is certainly not convex due to the discrete variables. In other words, thus, the multiple solutions of a set of HDAE are *local minima* – and so are the solutions of a set of nonlinear DAE – no matter how rigorous is the procedure to determine the switching of the discrete variables.

A consequence of the existence of (potentially many) feasible solutions leads is to a (probably unsolvable) implementation issue that is common to both time-stepping and event-driven approaches: the occurrence of untimely/unnecessary switches of discrete variables. The main issue is that a given event is identified by the change of sign in the vector $\boldsymbol{\varsigma}$ but the exact time at which such event happens is not known unless one solves a special problem called *zero-crossing problem*, which can be formulated as follows:

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{\varphi}(\boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{u}_*, t)\,, \\ 0 &= h_i(\boldsymbol{z}, \dot{\boldsymbol{z}}, t)\,, \end{aligned} \tag{49}$$

where h_i is the i-th event function that is binding and $\boldsymbol{u}_*$ is the vector of discrete variables that is *frozen* for the solution of the zero-crossing problem. This problem has $n + m + 1$ equations and $2n + m + 1$ unknowns, namely $\boldsymbol{z}$, $\dot{\boldsymbol{z}}$ and t. In stead of solving (49), thus, one can solve the following problem:

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{\chi}(\boldsymbol{z}, \boldsymbol{z}_t, \boldsymbol{u}_*, t)\,, \\ 0 &= h_i(\boldsymbol{z}, \dot{\boldsymbol{z}}, t)\,, \end{aligned} \tag{50}$$

where $\boldsymbol{\chi}$ is a function that depends on the numerical integration scheme (see for example [64]) and $\boldsymbol{z}_t$ is the value of the states at the previous step.

The solution of the zero crossing problem (50) tends to be time consuming and may require to have to cut the time step until the change of sign of only one element of $\boldsymbol{\varsigma}$ occurs. Due to its nonnegligible computational burden, the solution of a zero-crossing problem is generally the last resource that is utilised, for example, when the

time-stepping or event-driven approaches fail or produce chattering. In the software tool utilised in the simulations shown in Section 3, the time-stepping approach is utilised for most discontinuous right-hand side equations, whereas the event-driven approach is mostly utilised for behavioural models.

As a final note on time- and state-driven approaches, we cite the Quantised Integration Methods (QIMs) that had some hype about a decade ago [122]. These numerical methods work by quantising the increments of the states rather than time. The advantage is that the points that the integration method evaluates impose a given state, which make unnecessary the solution of a zero-crossing problem. In turn, the difference between a conventional time-driven integration method and QIMs is very similar to that that exists between the perpendicular intersection and the local parameterisation method utilised for the corrector step of a continuation power flow analysis [59, 123]. As any other method, QIMs have advantages and drawbacks. In the experience of the authors, the main issues with the QIMs are two: (i) no A-stable integration method equivalent to the ITM or the BDFs is available for QIMs; (ii) QIMs are really efficient only for linear circuit-like DAE and if not too many events occur in a simulation. But, of course, most integration methods are efficient in these conditions. In particular, QIMs become awfully slow for SDAEs, which has to be expected as continuous Wiener processes can be viewed as a sequence of infinitely many events in the unit of time. It is also for this reason that stochastic processes require *ad hoc* integration schemes.

2.3 Fractional-Order Differential-Algebraic Equations

The last specialised model that we discuss here considers Fractional-Order Differential-Algebraic Equations (FO-DAEs). Fractional calculus deals with the problem of the differentiation and integration operators d^γ/dt^γ, $\int_0^t d^\gamma(\tau)$ for $\gamma \in \mathbb{R}$. This is an example of advanced mathematical techniques and their applications to power system analysis and control that have been explored during the execution of the project AMPSAS.

The first problem to solve is which theoretical definition to use. There are in fact many different ways to define fractional derivatives [124–130] but only few of them are adequate for an implementation in a software tool. In this work, we consider the definition proposed by Caputo in [128], which reads:

$$x^{(\gamma)}(t) = \frac{d^\gamma x}{dt^\gamma} = \frac{1}{\Gamma(\mu-\gamma)} \int_0^t \frac{x^{(\mu)}(\tau)}{(t-\tau)^{\gamma-\mu+1}} d\tau \,. \tag{51}$$

The Laplace transform of (51) is:

$$\mathcal{L}\{x^{(\gamma)}(t)\} = s^\gamma X(s) - \sum_{j=0}^{\mu-1} s^{\gamma-j-1} x^{(j)}(0) \,. \tag{52}$$

Equation (52) requires the knowledge of the initial conditions $x^{(j)}(0)$, $j = 0, 1, \ldots, \mu-1$, which in this case are of integer order. This property effectively allows for the solution of initial value problems. Then, it remains to implement the term s^γ in (52). This is typically approximated using appropriate rational order transfer functions. Also in this case, various techniques are available, e.g., [131]. In practice, the most commonly utilised continuous method is the Oustaloup Recursive Approximation (ORA) [132]. The generalised ORA of a fractional derivative of order γ is defined in Laplace domain as [126]:

$$s^\gamma \approx \omega_h^\gamma \prod_{k=1}^{N} \frac{s+\omega_k'}{s+\omega_k} \,, \tag{53}$$

where $\omega_k' = \omega_b \omega_v^{(2k-1-\gamma)/N}$, $\omega_k = \omega_b \omega_v^{(2k-1+\gamma)/N}$, and $\omega_v = \sqrt{\omega_h/\omega_b}$, with $[\omega_b, \omega_h]$ being the frequency range for which the approximation is designed to be valid and N the order of the polynomial approximation. The block diagram of ORA is shown in Fig. 4. This is, for now, the implementation utilised in the software tool developed for the project AMPSAS but the implementation allows implementing any other approximation available in the literature.

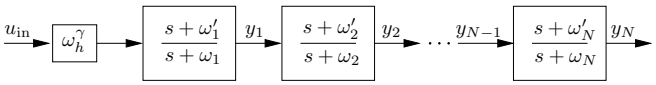


Fig. 4: Oustaloup's recursive approximation block diagram [65].

In the context of the project AMPSAS, an effort has been done to provide a comprehensive introduction to fractional order calculus for power system modelling and control and small-signal stability analysis in [65, 133–135]. In fact, while fractional order derivative do not have really a physical meaning, they find applications in control. Being ultimately converted into a series of leads-lags, from the control design point of view one has to define only very few parameters and the order of the fractional derivatives.

2.4 Proposed General Model

We are now ready to present the proposed general SFHI-DAE-based model for the transient stability analysis of power systems. This combines the implicit, frequency-dependent power system model with inclusion of stochastic processes and jumps, delays and discontinuous right-hand sides and behavioural models. Merging together (28), (29), (37) and (47), one obtains:

$$\begin{aligned} \mathbf{0} &= \boldsymbol{\varphi}(\boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{z_d}, \boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, \boldsymbol{u}, t) \,, \\ d\boldsymbol{\eta} &= \boldsymbol{a}(\boldsymbol{\eta}, t)dt + \boldsymbol{b}(\boldsymbol{\eta}, t) \odot d\boldsymbol{\zeta} + \boldsymbol{c}(\boldsymbol{\eta}, t) \odot d\boldsymbol{\kappa} \,, \\ \mathbf{0} &= \boldsymbol{k}(\boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{z_d}, \boldsymbol{u}, t) \,, \\ \boldsymbol{\varsigma} &= \boldsymbol{h}(\boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{z_d}, t) \,, \end{aligned} \tag{54}$$

where all variables and functions have the meaning indicated in the previous sections.

The implementation of a software tool that embeds all the features discussed so far and effectively implements (54) in a meaningful and consistent way has been a titanic task. This has required, in effect, the last 12 years of work of the first author and the combined effort of several students and post-doctoral researchers, not limited to the co-authors of this paper. The effort is so vast that the methodological approach adopted for the implementation of the code is as important as the theoretical foundations themselves. The key aspects of this methodological approach are briefly outlined in the reminder of this section.

2.4.1 Modularity: Modularity is one of the most important concepts of modern programming techniques. It is the foundation of object-oriented programming, servlets, multi-threading, and, more recently, micro-services. It is thus natural that modularity is also an important concept for the development of a software tool for power system simulation. In this context, modularity arises at two different levels. At the device level, one can expect that most device that compose a power system do not include all features that are accounted for in (54). Assuming that every device is implemented as a *class*[*], not all devices will include stochastic processes, delays and events. These features should thus conveniently implemented in separated methods and be called only if and when needed. A similar principle applies at the system level. The device models available in the libraries of the software tool are not always utilised altogether in simulations. The code of the devices that is not part of a system model should thus not be called during a simulation. This concept is called *just in time* (jit) compilation and belongs to the more general concept of *laziness*, which is a virtue when it refers to computer

[*] *The interested reader can refer to [136] for a gentle introduction to object-oriented programming for scientific computing.*

programming as it increases the efficiency and performance of the code.

2.4.2 Acausality: Acausality is a fundamental concept of physical system modelling. It refers to the fact that the quantities that appears in a physical law should not be interpreted as one the cause of another, but a condition that each quantity contributes to satisfy [137]. This concept is better explained through an example. Let us consider again the double lead-lag shown in Fig. 2. Control theory suggests that u is the input and y is the output, thus implicitly assuming that u *causes* y. But this interpretation is not included in the equations that define the transfer function of the double lead-lag and should not affect how these equations are written. The implicit formulation helps foster acausality when writing DAEs. For example, the equations (16) of the double lead-lag of Fig. 2 can be rewritten as:

$$\begin{aligned} 0 &= T_2\dot{x}_1 + x_1 - u\,, \\ 0 &= T_4\dot{x}_2 - T_1\dot{x}_1 + x_2 - x_1\,, \\ 0 &= T_3\dot{x}_2 + x_2 - y\,, \end{aligned} \tag{55}$$

which are perfectly symmetric and do not allow distinguishing between inputs and outputs.

2.4.3 Separation of the solvers from the power system model: All algorithms and techniques implemented throughout the execution of the project will be defined in general terms, i.e., without any particular device or system model in mind. This approach is recognised as a necessary feature of large-scale projects that involve the simulation of physical systems [54]. Direct advantages of this approach is that the implementation of the algorithm can be conducted in parallel with and independently from the implementation of physical models. Another major advantage is that no hypothesis is imposed on the device models, thus allowing an unconstrained development of the system model. This approach is seldom applied to power system analysis. A relevant example of an algorithm that heavily constrains device models is the well-known fast decoupled power flow analysis, which is certainly efficient, but can be applied only to a specific (and very restrictive) model of the transmission system, loads and generators [138].

2.4.4 Focus on large-scale systems and real-world data: Every technique and algorithm should be tested using "large-scale power systems". This point is crucial as the computational burden tends to grow quickly when dealing with nonlinear systems. Any technique that does not prove efficient or scalable will be discarded *a priori*. On the other hand, parallelisation will be exploited whenever possible. Moreover, it is crucial to base and test all techniques on a real-world system. The case study presented in Section 3 is an application of this methodological principle.

2.4.5 High performance computing: The model (54) proposed in this paper can have a heavy computational burden, especially for large systems. Thus, the implementation of algorithms and techniques based on high performance computing and, in particular, parallel programming that are able to exploit multi-threaded computer architectures is a key aspect of the proposed model. Parallel programming for power system analysis is a relatively recent topic [139–146], although there are also pioneering works dated back in the late 70s [147]. Needless to say, the simulation proposed model (54) can significantly benefit from parallelisation. An obvious application is the study of the effects of stochastic processes through a Monte Carlo method. This is an *obvious* parallelisation as each realisation of the processes and each simulation of the Monte Carlo analysis are fully independent and can be thus solved on different processors without any particular coordination [86]. Another, less trivial, example of technique that can foster code parallelisation is the one-step delay decoupling technique described in [103].

3 Case Study

This section illustrates the dynamic performance of the proposed SFHI-DAE model through a variety of examples based on the All-Island Irish Transmission System (AIITS). The base-case scenario of this grid consists of 1,479 buses, 1,851 transmission lines, 245 loads, 22 conventional synchronous power plants with Automatic Voltage Regulators (AVRs) and turbine governors, 6 PSSs, one Automatic Generation Control (AGC) that coordinates the synchronous machines, and 176 wind power plants.

The topology and the steady-state operation data of the grid were provided by the EirGrid Group, the Irish TSO (see Fig. 5). Dynamic data, on the other hand, are defined based on the technology of the generators and do not represent any actual operating condition. We have, however, duly tested the dynamic response of the base-case scenario in order to match, at least in the first second after a large contingency, that of the real-world system [148, 149]. Figure 6 shows the dynamic response as measured by the EirGrid Group as well as the system setup for the AMPSAS project following the outage of the largest in-feed, namely the East West Interconnector (EWIC) occurred on the 28th of February 2018. On this date, the VSC-HVDC link EWIC [20] that connects the AIITS with the Great British transmission system (GB), was tripped. At that moment, the AIITS was exporting 470 MW to GB. Due to the loss of the EWIC, the frequency in the Irish grid rose to 50.42 Hz. Over frequency protections were triggered and several wind farms were curtailed.

The examples below were developed during the execution of the project AMPSAS and illustrate the impact on the AIITS dynamic performance of frequency-dependent models (Section 3.1); correlated stochastic processes of wind speeds (Section 3.2); WAMS communication delays (Section 3.3); PI controller limiters on the VSCs of the EWIC (Section 3.4); and a fractional order AGC (Section 3.5). All simulations were carried out using the software tool Dome [55].

Verification of the results has been obtained by cross checking all results and running a large number of sensitivity studies. We observe that, besides the AIITS, we have also thoroughly tested the models proposed in the manuscript with a large variety of test and benchmark systems. The interested reader can find additional simulation results and tests in the various references emanated by the project AMPSAS and duly cited in the following sections.

3.1 Frequency Dependent Model

This example is taken from [76] and illustrates the relevance of considering the frequency dependency of the device models in a low-inertia system. Figure 7 shows the transient response of the AIITS following the outage of the EWIC. This contingency is chosen on purpose as it is the most severe power unbalance that can be triggered by a single event in the AIITS and, thus, leads to significant frequency variations.

The notation utilised in Fig. 7 is the following:

- The Transient Stability Model (TSM) considers constant reactances and susceptances everywhere in the grid. This is effectively the conventional model as in (1).
- The Frequency Dependent Model (FDM), which was proposed in [76] and is discussed in Section 2.2.1, considers the dependency on the frequency in all loads, branches and generators. In particular loads are modelled as steady-state full-load induction motors [150].
- A balanced fundamental-frequency Dynamic Phasor Model (DPM) – based on Park vectors – that includes machine flux and line dynamics.

The trajectories shown in Fig. 7 indicate that the DPM and the FDM show similar information. The DPM also show some fast flux and electromagnetic dynamics but these damp quickly and do not modify the overall behaviour of the grid, as expected. On the other hand, the TSM is conservative, as it shows a frequency nadir that is about 100 mHz lower and larger voltage variations than those obtained with the other two system models. From a computational point of view,

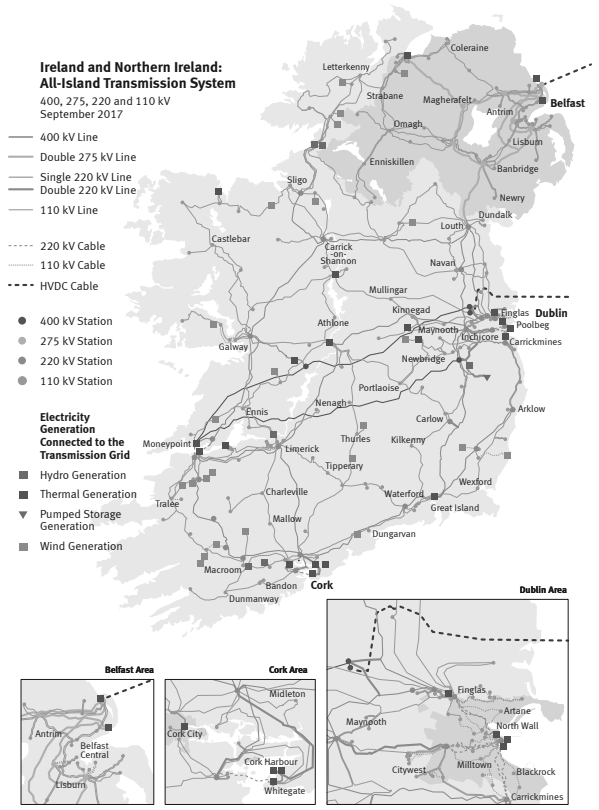


Fig. 5: Scheme of the All-Island Irish Transmission System (courtesy of EirGrid Group, available at www.eirgridgroup.com).

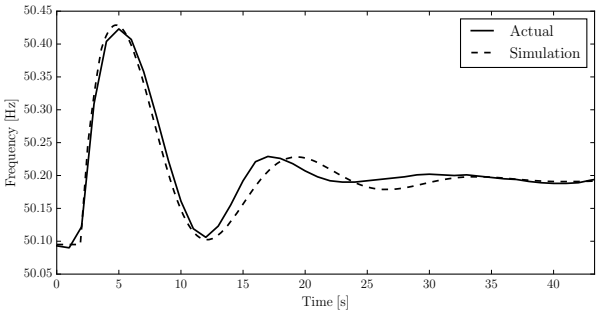


Fig. 6: Frequency response of the AIITS following the outage of the largest in-feed [149].

the TSM, FDM and DPM take 14, 7.5, and 165 s, respectively, to complete the simulations. The bigger time required by the DPM is due to its heavier computational burden and to the need to use a smaller time step (0.002 s vs. 0.01 utilised for the other two models) needed to account for fast electromagnetic dynamics.

We note that these results have been "dramatised" by the utilisation of load models that heavily depend on voltage and frequency variations. Without this dependency, the differences between the TSM and the other models reduce significantly. On the other hand, one may argue that in a scenario with extremely low or even zero inertia, frequency variations can be much higher than those that are possible in the current system. It has to be expected, thus, that the relevance of frequency dependent models will increase in the near future.

3.2 Correlated Stochastic Processes

This example, which is taken from [76], illustrates the impact of volatility and correlation of stochastic perturbations of loads, bus voltages and wind speeds on the dynamic performance of the AIITS. We first group the loads and wind power plants of the AIITS into

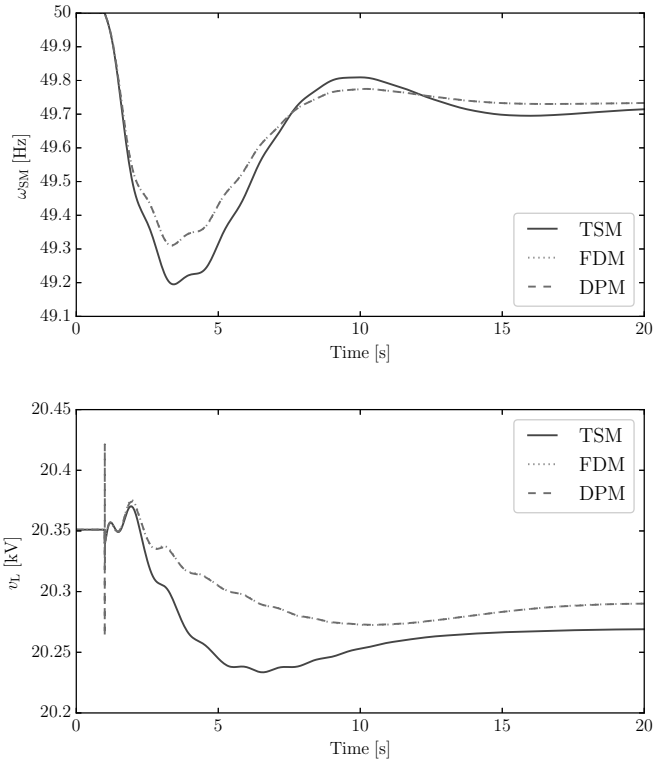


Fig. 7: Rotor angular speed of a synchronous machine (top panel) and voltage magnitude at a load bus (lower panel) of the AIITS following the outage of the largest in-feed [76].

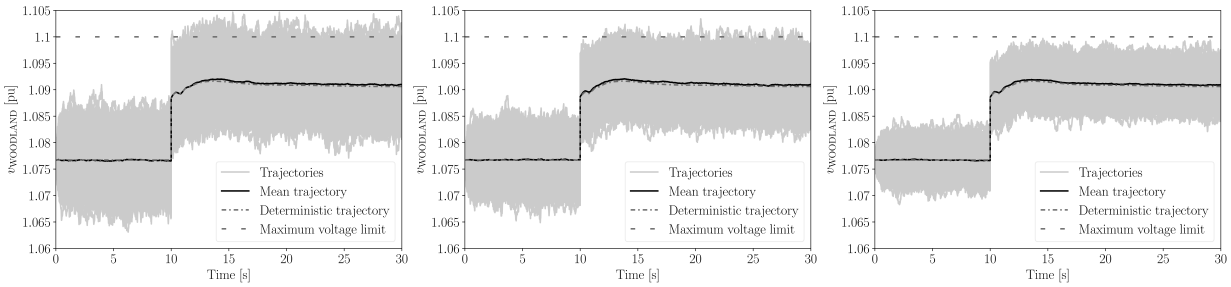
areas and then consider three scenarios with same properties of the stochastic processes except for the correlation, as follows.

- S1 considers model (21), i.e., a scenario where all Wiener processes are fully uncorrelated.
- S2 considers model (28), with a low level of correlation among processes that belong to the same area, i.e. $R_{i,j} = 0.4$.
- S3 considers model (28) with a high level of correlation among processes that belong to the same area, i.e. $R_{i,j} = 0.8$.

In all scenarios, the elements of the correlation matrix for i and j belonging to two different areas are assumed to be $R_{i,j} = 0$. Finally, the contingency is, also in this example, the disconnection of the EWIC, which occurs at $t = 10$ s.

Figures **??**, **??** and **??** show the voltage magnitude at bus-bar Woodland obtained with 1,000 Monte Carlo simulations, for S1, S2 and S3, respectively. The black solid lines represent the mean value of the 1,000 trajectories. Since the stochastic processes are designed to have zero expectation, the black lines represent the dynamic response of the deterministic case, i.e., the case for which the diffusion terms of models (21) and (28) are $\boldsymbol{b} = \mathbf{0}$. While the expected value is always within the voltage limits (indicated with the dashed line), depending on the correlation of the processes, each scenario shows a different probability of violating this limit, namely 24% for S1, 7% for S2 and never for S3 in the interval $t \in [10, 30]$ s.

We have selected on purpose this example as it is fully counter-intuitive and shows results that are opposite to the expected ones. In most cases, in fact, the higher the correlation among the processes, the higher the probability to drive the system to instability (see for example [95]). This can be readily explained considering load consumption: if two loads are strongly correlated, their energy consumption, if it increases, will increase at the same time, thus reducing the loading margin of the system. On the other hand, the variations of fully uncorrelated loads will more likely average out. In the simulations shown in Fig. 8, however, exactly the opposite behaviour is observed: the higher the correlation the lower, statistically, the variations of the voltage. These results could not predicted



(a) No correlation (b) Low correlation (c) High correlation

Fig. 8: Impact of stochastic processes on the voltage magnitude of the bus-bar Woodland [76].

a priori and are due to the combined effect of correlated stochastic wind generation and correlated load variations.

A relevant observation, that is common to all works on SDAEs that were carried out during the execution of the project AMPSAS, is the difficulty to find suitable measurement data to set up realistic stochastic processes. In fact, the parameters of the stochastic processes included in model (54) can be determined only through long time series with small sampling rate at several different locations. The lack of these data is, we believe, due to two concurrent factors. On the one hand, these measurements have not been deemed necessary (or even possible) by TSOs and DSOs until recent years, i.e., until the level of penetration of renewable resources such as wind and solar generation has significantly increased in power systems around the world. On the other hand, SDAEs and their often unintuitive behaviour are not known to most practitioners and academics. As far as we know, in fact, SDAEs are not thought in undergraduate or graduate programmes on electrical engineering. We see, thus, the need for updating electric power engineering programmes to include this important and timely subject.

3.3 Impact of WAMS Delays

This example, extracted from [104], assumes that the 6 PSSs of the AIITS are included in a WAMS. Thus, their input signals of the PSSs are affected by time-varying delays similar to that shown in Fig. 3. In this example, the contingency consists in the outage of the synchronous power plant connected to bus 1378. To better illustrate the effect of the delays, we assume that the PSSs have relatively high gains. Without delays or with delays but with low PSS gains, in fact, the system is stable and performs well.

Figure 9 shows the results obtained for three scenarios: no delays, constant delay $\bar{\tau}$ and stochastic WAMS delay. The constant delay is assumed to be the average value of the stochastic one. The effect of the high gains of the PSSs is to increase the damping of electromechanical oscillations but also, and as expected, to make the PSSs more sensitive to measurement delays. This results in a small limit cycle when considering the constant delay. The WAMS delays, however, has dramatic consequences on the dynamic performance of the system.

While this result is obtained by pushing the gain of the PSSs, yet it is a relevant demonstration of the effects, on a real-world system, of the quenching phenomenon. This example is also particularly interesting because it combines delays, stochastic processes and discontinuities (given by the sawtooth waveform for the WAMS communication system). It is thus an example that could not be carried out without the implementation of the proposed SFHI-DAE model.

3.4 Impact of PI Limiters on the EWIC

This example, taken from [118], illustrates the impact of different implementations of the hard limits of the current controllers of the VSCs of the EWIC. In this example, thus, we include a dynamic model of the EWIC, which is modelled as a symmetric monopole-type VSC as described in [151], and a simplified 63-bus dynamic model of the GB system as described in [152]. Before the fault, the

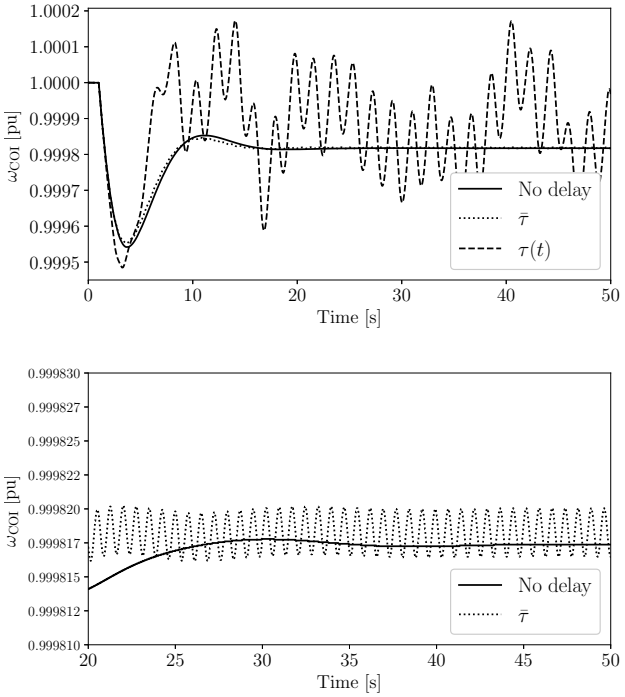


Fig. 9: Transient behaviour of the frequency of the center of inertia (COI) for the AIITS following a power plant outage [104].

EWIC imports 450 MW from the GB to the AIITS system. The contingency is a three-phase fault located close by the EWIC on the Irish side. The fault occurs at 0.2 s and is cleared after 60 ms.

Figure 10 shows the trajectories of the reference dq-axis currents of the inner control loop of the Irish-side VSC. These controllers are implemented as PIs and are utilised to impose the active and reactive power flow, respectively, in the EWIC. In [118], several models are considered, as follows. PI0 is a plain PI with no limiters, which is utilised for reference; PI1 includes a windup limiter; and PI2 to PI6 include different implementations of anti-windup limiters. These are: IEEE Standard 421.5-2016 with conditional integrator (PI2); two types of back calculation (PI3 and PI4); back calculation with delay (PI5); and combined conditional and back calculation (PI6).

During the fault, all PI implementations, except for PI0, always reach their limits on both d- and q-axis. As expected the windup limiter (PI1) is the slowest to recover as it does not lock its internal state when the limiters are enabled. The other limiters respond faster, in particular PI2 the fully locks its internal state. Back calculation models PI3 to PI6 are also anti-windup but do not lock their states, rather use a feedback signal to reset this state and prevent the integrator from winding up. Due to the relatively short duration of the fault, the integrators of the PI models 3 to 6 do not reach a steady state and provide a different dynamic response that depends on their specific implementation.

As for the stochastic and delay DAEs examples, the main difficulty to set up realistic scenarios and simulations is the availability of data. VSC devices and their controllers are often provided as *black*

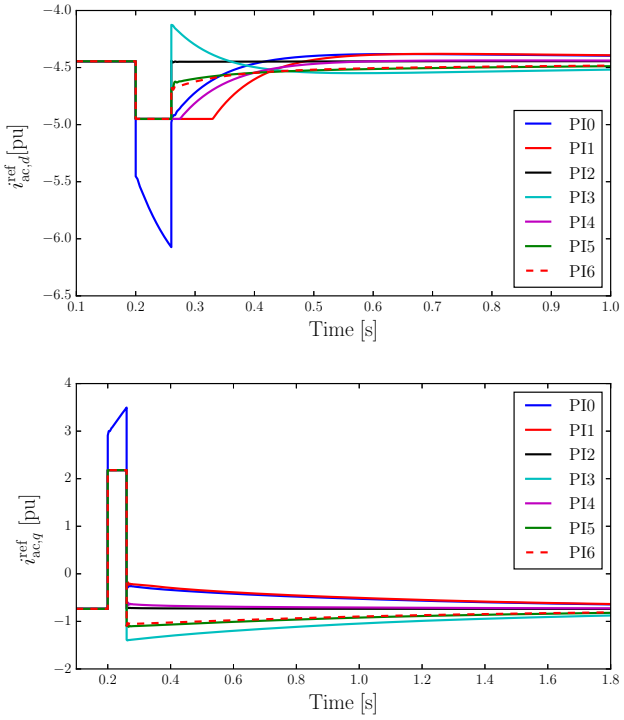


Fig. 10: Dynamic response of the VSC reference dq-axis currents of the active and reactive power support on the Irish-side of the EWIC [118].

boxes by the makes and the details of the implementation of the hard limits are not disclosed or are difficult to figure out. As a matter of fact, in many studies that can be found in the literature, such limiters are not modelled at all. However, as the results shown in this example indicate, small differences in the implementation of the hard limits can make quite a significant difference in the output of the controllers and, in turn, in the power system dynamics. We believe, thus, that creating a consciousness on the importance of modelling precisely the models of the VSC controllers is critical, especially in view of the fact that VSC-connected devices will dominate power system dynamics in the near future.

3.5 Fraction Order AGC

This final example, extracted from [65], illustrates the utilisation of a fractional order controller for the AGC of the AIITS. Figure 11 shows the frequency response of the AIITS with a conventional integer order integral controller (IO-AGC) and the fractional order integral one (FO-AGC). The gain of both controllers is $K_i = 500$. Then, for the FO-AGC, the fractional order is chosen as $\gamma = 0.15$ and the ORA parameters are $[\omega_b, \omega_h] = [10^{-3}, 10^{1}]$ rad/s, $N = 4$. Finally, we apply the same contingency, i.e., EWIC outage, as that the leads to obtain Fig. 6. Simulation results show that the FO-AGC can improve, even though slightly, the frequency regulation of the system. This, of course, is a simple example, but shows how inexpensive controller setups can make a difference on the dynamic performance of a real-world network.

4 Conclusions

This paper discusses the methodological and numerical challenges of the implementation of a software tool for power system modelled as a set of SFHI-DAE. The modelling and applications of stochastic processes, delays, discontinuities and behavioural models to power system analysis are discussed in the paper. Stochastic processes serve to model renewable sources and loads. Delayed and hybrid (e.g., discontinuous) DAEs are utilised to model imperfections and deviations from the ideal behaviour of telecommunication systems, measurement signals and centralised/distributed controllers. Signal

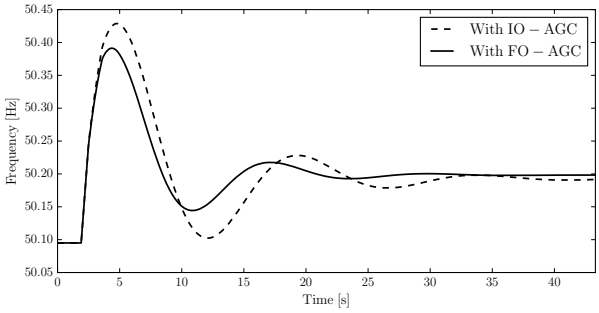


Fig. 11: Impact of FO-AGC on frequency response [65].

transmission delays, digital discretisation, noises and information loss are also considered.

Each considered modelling aspect is particularly challenging as it involves merging advanced mathematical concepts that are currently not used for power system analysis with an expert knowledge of computer-based modelling and simulation techniques. The result is a sophisticated tool that allows solving complex stability analysis and control design problems.

The discussions and examples presented in the paper show that it is possible for a small team of researchers with the adequate interdisciplinary skills and to develop a complete software tool based on advanced mathematical concepts. It just requires a long time: 5 years for the execution of the project AMPSAS plus the previous 15 years of experience of the first author and lead of the project.

Arguably, the most relevant conclusions and recommendations that can be drawn are two: (i) the importance of modelling "details," also at the cost of significantly complicate the power system model and its implementation in a computer language; and (ii) the crucial importance of real-world data to validate and set up such models. These two aspects often go together: it is hard to justify the need for certain measurements if there is no software that can actually give them a use. On the other hand, it is risky to commit a research group to the study of certain modelling details that nobody has studied yet. There is in fact the probability that these "details" do not actually have an impact on the dynamic behaviour of the system. As a matter of fact, what has not been discussed in this paper are the several tens of attempted implementions of model features that served only to realise that these features have no particular impact on the system dynamic performance. Nevertheless, the lesson learnt is that this kind of studies, even if unconventional and not following the main-stream research trends, are useful and definitively worth trying.

We believe that the discussions and results presented in this paper provide a novel perspective and a novel *methodological* approach to the formulation and analysis of power systems. We also trust that the proposed modelling approach can provide an improved flexibility and capability of adequately reflecting the fundamental changes that power systems has gone through in recent years, and will continue to go through into the future.

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