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Fractional–order dynamical model for electricity markets

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Summary

In this article we use a generalized system of differential equations of fractional order to incorporate memory into an electricity market model. By using this idea, essential information from the past, such as the behavior of market participants, namely suppliers and consumers, can be used and have impact on future decisions. We construct the fractional–order dynamical model, study its solutions, and provide closed formulas of solutions. Finally, we provide an application by using the proposed formula of solutions as well as a numerical example which also compares the proposed model with a conventional, integer-order electricity market model. Results indicate that the inclusion of memory leads market participants to adopt a conservative behaviour.

KEYWORDS:

Singular systems, Caputo, fractional, electricity market

1 | INTRODUCTION

Electricity markets are essential tools due to the fact that they offer flexibility to a power system by maintaining the power balance until physical generation or consumption.

With the rise of several renewable sources such as wind and solar energy, see [1], the importance of studying and go deeper into electricity market models has become more important than ever.

It can be said that time scale of electricity markets is similar to long-term power system dynamics like secondary frequency control [2]. However, this similarity on the timescales causes also a concern on the coupling between the dynamic response of the power system and electricity markets, see [3,4,5].

The model described in the present work is a generalized system of differential equations of fractional order. It is highly realistic since it succeeds to describe the dynamic aspects of systems and incorporate the desired memory into the electricity market model by adding the essential information how the memory of market participants, namely suppliers and consumers, impacts on their behavior, i.e., on their bids. Thus it incorporates more of the factors that determine the model than any of the previous works seen in the literature. In other words, taking into account the memory of market participants is of utmost importance in economic processes as they can remember the changes of economic indicators and factors in the past [6]. These changes can then impact their behaviour and decisions.

Generalized systems of differential equations, see [7,8,9,10,11,12], and difference equations, see [13,14,15] have attracted the interest of several researchers in the last few decades. Some interesting results have also been obtained for singular systems of equations evolving fractional operators, see [16,17,18,19,20,21,22,23,24]. A generalized system of linear differential equations has the form:

$$\boldsymbol{E}\,\boldsymbol{x}'(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{\omega}(t)\,, \tag{1}$$

where $\boldsymbol{E}, \boldsymbol{A} \in \mathbb{C}^{r\times m}$, $\boldsymbol{x} : [0,+\infty) \to \mathbb{C}^{m\times 1}$, $\boldsymbol{\omega} : [0,+\infty) \to \mathbb{C}^{r\times 1}$. The matrices $\boldsymbol{E}$, $\boldsymbol{A}$ can be non-square ($r \neq m$), or, square ($r = m$) with $\boldsymbol{E}$ regular ($\det\boldsymbol{E} \neq 0$) or singular ($\det\boldsymbol{E}$=0). In the case that the matrices are non-square or square with $\boldsymbol{E}$ singular we will refer to (1) as a singular system. In the case that the matrices are square with $\boldsymbol{E}$ regular we will refer to (1) as a regular system.

The pencil of a regular system has finite eigenvalues, while the pencil of a singular system has additional type of invariants, an infinite eigenvalue in the case of a regular pencil, see[21], and in addition row-column minimal indices in the case of a singular pencil, see[12]. This type of systems appear in control theory, see[25,26,27], and in several applications in electrical engineering such as the modeling of electrical circuits, see[11], and power system dynamics, see[28,29,30,31]. Despite several studies, most articles deal with regular systems and avoid the case of singularities, a case that is also included in this article.

The paper is organized as follows. Section 2 contains the description of the proposed model, a system of fractional differential equations equations governing the whole model. In section 3 we study the solutions of the system and provide closed formulas of solutions. Section 4 contains an example using the obtained formula of solutions, and a practical application that provides further insight and better understanding as regards the control actions, system design by using a special and realistic case of the fractional order dynamical system. Section 5 concludes the entire paper.

2 | THE MODEL

The original version of Alavarado's model proposes a dynamic market model to study the couplings between the dynamics of the power network and the short-term electricity market, see[4]. It is based on the following equations:

- The first equation accounts for the system power imbalance indirectly, i.e., through the deviation frequency of the Center-of-Inertia (CoI) with respect to the reference frequency:
$$T_\lambda \frac{d\lambda(t)}{dt} = -H_d\lambda(t) + K_E(\omega^{\mathrm{ref}} - \omega_{\mathrm{CoI}}(t)), \tag{2}$$
where $\frac{d\lambda(t)}{dt}$, $\lambda(t)$ is the marginal electricity price, and the electricity price respectively; ω^{ref} represents the reference frequency; $\omega_{\mathrm{CoI}}(t)$ represents the frequency of the CoI, i.e. $\omega^{\mathrm{ref}} - \omega_{\mathrm{CoI}}(t)$ is the deviation frequency of the CoI with respect to the reference frequency; T_λ is the time constant; H_d is the deviation with respect to a perfect tracking integrator and for a Low-Pass Filter (LPF) it is $H_d = 1$; and K_E can be written as $K \cdot \lambda(t)$ and be used as feedback gain.

- The second equation assumes that a generator will increase its power production if the electricity price is higher than its marginal cost:
$$T_{gi}\frac{d\Delta P_{gi}(t)}{dt} = \lambda(t) - c_{gi}\Delta P_{gi}(t) - b_{gi}, \tag{3}$$
where $\Delta P_{gi}(t)$ is the generator active power; c_{gi}, b_{gi} are the parameters of the marginal cost and benefit of the generator, respectively; and T_{gi} is the time constant;

- The third equation assumes that a load will decrease its power consumption if the electricity price $\lambda(t)$ is higher than its marginal benefit.
$$T_{di}\frac{d\Delta P_{di}(t)}{dt} = -\lambda(t) + c_{di}\Delta P_{di}(t) + b_{di}, \tag{4}$$
where $\Delta P_{di}(t)$ is the load active power; c_{di}, b_{di} are the parameters of the marginal cost and benefit of the load, respectively; T_{di} is the time constant.

If one assumes loads to be inelastic (i.e., not considering (4)), then the market model (2)-(3) has a very similar structure to that of a conventional secondary frequency control, i.e. the automatic generation control (AGC)[27]. To better illustrate similarities, the control diagrams of a conventional AGC and that of the market model (2)-(3) (or MAGC) are depicted in Fig. 1 and Fig. 2 , respectively. It can be seen that the input of both controllers is the same. The AGC includes an integrator with gain K_o that has a similar function with the LPF block of the market, namely, to reduce the frequency oscillations. Finally, the outputs of the AGC and MAGC are distributed to the turbine governors (TGs) of the synchronous generators proportionally to their droops (R_i) and bids, respectively.

Equations (2)-(3) can be written as a matrix equation and form the generalized system of differential equations (1) with
$$\boldsymbol{E} = \begin{bmatrix} T_\lambda & 0 & 0 \\ 0 & T_{gi} & 0 \\ 0 & 0 & T_{di} \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} -H_d & 0 & 0 \\ 1 & -c_{gi} & 0 \\ 0 & 0 & c_{di} \end{bmatrix}, \quad \boldsymbol{x}(t) = \begin{bmatrix} \lambda(t) \\ \Delta P_{gi}(t) \\ \Delta P_{di}(t) \end{bmatrix},$$

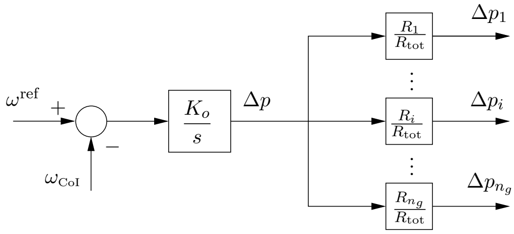

FIGURE 1 AGC control diagram.

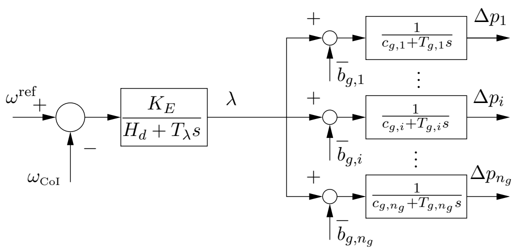

FIGURE 2 MAGC control diagram.

and

$$\boldsymbol{B} = \begin{bmatrix} \omega^{\text{ref}} - \omega_{\text{CoI}}(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} K_E \\ -b_{gi} \\ b_{di} \end{bmatrix}, \quad \boldsymbol{\omega} = \boldsymbol{B}\boldsymbol{u}.$$

Next we will define the Caputo fractional derivative that we will use as tool for our model.

Definition 1.1. (see [19], [21]) Let $Y : [0, +\infty) \to \mathbb{R}^{m\times 1}$, $t \to Y$, denote a column of continuous and differentiable functions. Then, the Caputo (C) fractional derivative of order a, $0 < a < 1$, is defined by:

$$\boldsymbol{Y}_C^{(a)}(t) := \boldsymbol{Y}^{(a)}(t) = \frac{1}{\Gamma(1-a)} \int_0^t \left[(t-x)^{-a}\boldsymbol{Y}'(x)\right] dx \, .$$

In order to simply explain why the proposed fractional derivative and its memory effect will relate to our model, we will use the discrete version of (1). An alternative way to represent this system, formed through (2)-(4), is the following generalized discrete time system:

$$\boldsymbol{E}\,\boldsymbol{x}_{k+1} = \boldsymbol{A}\,\boldsymbol{x}_k + \boldsymbol{\omega}_k, \quad k \in \mathbb{N}. \tag{5}$$

Where

$$\boldsymbol{E} = \begin{bmatrix} T_\lambda & 0 & 0 \\ 0 & T_{gi} & 0 \\ 0 & 0 & T_{di} \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} -H_d & 0 & 0 \\ 1 & -c_{gi} & 0 \\ 0 & 0 & c_{di} \end{bmatrix}, \quad \boldsymbol{x}_k = \begin{bmatrix} \lambda_k \\ \Delta P_{gi\,k} \\ \Delta P_{di\,k} \end{bmatrix}.$$

and

$$\boldsymbol{B} = \begin{bmatrix} \omega^{\text{ref}} - \omega_{\text{CoI}}(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} K_{E\,k} \\ -b_{gi} \\ b_{di} \end{bmatrix}, \quad \boldsymbol{\omega} = \boldsymbol{B}\boldsymbol{u}_k.$$

Equation (5) is a first-order matrix difference equation. It cannot account for the memory of the market participants. The term $\boldsymbol{x}_{k+1}$ is only related to just a previous step in time namely the term $\boldsymbol{x}_k$. Hence, when using (5), we obtain the values of λ_{k+1}, $\Delta P_{gi\,k+1}$, $\Delta P_{di\,k+1}$ by only absorbing information from just a previous step in time k, and not considering all the "history"' of changes at times $k-1$, $k-2$, ..., k_0, where k_0 the initial time step which can be assumed zero, i.e. $k_0 = 0$. To include the information from all these time steps we will use the fractional nabla operator.

To define this fractional operator and how it is formed, we initially have to to define the backward difference operator of first order, denoted by ∇ (nabla operator), which when applied to a vector of sequences $\boldsymbol{Y}_k : \mathbb{N} \to \mathbb{C}^m$ it produces the following result:

$$\nabla \boldsymbol{Y}_k = \boldsymbol{Y}_k - \boldsymbol{Y}_{k-1};$$

while the backward difference operator of second order, denoted by ∇^2, is defined by:

$$\nabla^2 \boldsymbol{Y}_k = \nabla(\nabla \boldsymbol{Y}_k) = \boldsymbol{Y}_k - 2\boldsymbol{Y}_{k-1} + \boldsymbol{Y}_{k-2};$$

Similarly, the ν^{th} backward difference operator, ∇^ν, is defined by:

$$\nabla^\nu \boldsymbol{Y}_k = \frac{1}{\Gamma(\nu+1)} \sum_{j=0}^{\nu} (-1)^j \frac{1}{\Gamma(j+1)\Gamma(\nu-j+1)} \boldsymbol{Y}_{k-j}, \quad \nu \in \mathbb{N}.$$

Where $\Gamma(\cdot)$ is the Gamma function. In order to define the fractional nabla operator, see[20], we set:

$$\nabla^\nu \boldsymbol{Y}_k = \boldsymbol{f}_k.$$

Where $\boldsymbol{f}_k$, known vector of sequences. By solving for $\boldsymbol{Y}_k$ we get:

$$\boldsymbol{Y}_k = \frac{1}{\Gamma(\nu)} \sum_{j=\alpha}^{k} (k-j+1)^{\overline{\nu-1}} \boldsymbol{f}_j = \nabla^{-\nu} \boldsymbol{f}_k.$$

Where $b^{\bar{c}} = \frac{\Gamma(b+c)}{\Gamma(b)}$. Based on this expression, i.e. $\nabla^{-\nu} \boldsymbol{f}_k = \frac{1}{\Gamma(\nu)} \sum_{j=\alpha}^{k} (k-j+1)^{\overline{\nu-1}} \boldsymbol{f}_j$, if we define $\mathbb{N}_\alpha$ by $\mathbb{N}_\alpha = \{\alpha, \alpha+1, \alpha+2, ...\}$, α positive integer, and n fractional then the nabla fractional operator of n-th order for any $\boldsymbol{Y}_k : \mathbb{N}_a \to \mathbb{C}^m$ is defined by:

$$\nabla_\alpha^{-n} \boldsymbol{Y}_k = \sum_{j=\alpha}^{k} b_{k-j} \boldsymbol{Y}_j,$$

where $b_{k-j} = \frac{1}{\Gamma(n)}(k-j+1)^{\overline{n-1}}$, $j = \alpha, \alpha+1, ..., k-1, k$.

As already written, one has to consider not only one time step to absorb information from the past but also the "history" of changes throughout the timeline $0, 1, ..., k-1, k$. This should be applied to three equations that form system (5) and have a different effect in each case:

$$T_\lambda \lambda_{k+1} = \sum_{j=0}^{k} \gamma_{1,k-j} \left\{ -H_d \lambda_j + K_E(\omega^{\text{ref}} - \omega_{\text{CoI}j}) \right\}$$

$$T_{gi} \Delta P_{gi\,k+1} = \sum_{j=0}^{k} \gamma_{2,k-j} \left\{ \lambda_j - c_{gi} \Delta P_{gi\,j} - b_{gi} \right\}$$

$$T_{di} \Delta P_{di\,k+1} = \sum_{j=0}^{k} \gamma_{3,k-j} \left\{ -\lambda_j + c_{di} \Delta P_{di\,j} + b_{di} \right\},$$

where $\gamma_{i,k-j}$, $i = 1, 2, 3$, represents the memory functions. Assuming a power-law fading memory, the functions $\gamma(t)$ can be written as follows:

$$\gamma_{i,k-j} = \frac{1}{\Gamma(n_i)} (k-j+1)^{\overline{n_i-1}}, \quad j = 0, 1, ..., k-1, k,$$

where $\Gamma(n_i)$ are gamma functions; Equivalently we then have:

$$\begin{aligned} T_\lambda \nabla_\alpha^{n_1} \lambda_{k+1} &= -H_d \lambda_k + K_E(\omega^{\text{ref}} - \omega_{\text{CoI}k}) \\ T_{gi} \nabla_\alpha^{n_2} \Delta P_{gi\,k+1} &= \lambda_k - c_{gi} \Delta P_{gi\,k} - b_{gi} \\ T_{di} \nabla_\alpha^{n_3} \Delta P_{di\,k+1} &= -\lambda_k + c_{di} \Delta P_{di\,k} + b_{di}. \end{aligned} \tag{6}$$

Where $0 \leq n_i \leq 1$ are the fractional-orders of the nabla discrete operator; Returning to the continues time system (1), and by using the previous discussion, we propose the following fractional-order version of the dynamic electricity market model:

$$T_\lambda \lambda^{(n_1)}(t) = -H_d \lambda(t) + K_E(\omega^{\text{ref}} - \omega_{\text{CoI}}(t)), \tag{7}$$

$$T_{gi} \Delta P_{gi}^{(n_2)}(t) = \lambda(t) - c_{gi} \Delta P_{gi}(t) - b_{gi}, \tag{8}$$

$$T_{di} \Delta P_{di}^{(n_3)}(t) = -\lambda(t) + c_{di} \Delta P_{di}(t) + b_{di}, \tag{9}$$

where $0 \leq n_i \leq 1$ are the orders of the fractional derivatives. It's matrix form is:

$$\boldsymbol{E}\, \boldsymbol{x}^{\Xi}(t) = \boldsymbol{A}\, \boldsymbol{x}(t) + \boldsymbol{\omega}(t), \quad \boldsymbol{x}^{\Xi} = \begin{bmatrix} \lambda^{(n_1)}(t) \\ \Delta P_{gi}^{(n_2)}(t) \\ \Delta P_{di}^{(n_3)}(t) \end{bmatrix}. \tag{10}$$

Where $\boldsymbol{E}$, $\boldsymbol{A}$, $\boldsymbol{x}(t)$, $\boldsymbol{B}$, $\boldsymbol{u}$ as defined in (1). The pencil of the system is equal to, see[26]:

$$\begin{bmatrix} s^{n_1} & 0 & 0 \\ 0 & s^{n_2} & 0 \\ 0 & 0 & s^{n_3} \end{bmatrix} \boldsymbol{E} - \boldsymbol{A} = \begin{bmatrix} s^{n_1} & 0 & 0 \\ 0 & s^{n_2} & 0 \\ 0 & 0 & s^{n_3} \end{bmatrix} \begin{bmatrix} T_\lambda & 0 & 0 \\ 0 & T_{gi} & 0 \\ 0 & 0 & T_{di} \end{bmatrix} - \begin{bmatrix} -H_d & 0 & 0 \\ 1 & -c_{gi} & 0 \\ -1 & 0 & c_{di} \end{bmatrix} = \begin{bmatrix} s^{n_1} T_\lambda + H_d & 0 & 0 \\ -1 & s^{n_2} T_{gi} + c_{gi} & 0 \\ 1 & 0 & s^{n_3} T_{di} - c_{di} \end{bmatrix}.$$

The determinant of the pencil is equal to $(s^{n_1} T_\lambda + H_d)(s^{n_2} T_{gi} + c_{gi})(s^{n_3} T_{di} - c_{di})$ which means that the pencil of this system is regular, though the system can be singular if at least one of the elements T_λ, T_{gi}, T_{di} is zero or tends to be close to zero.

Note that the proposed model represents real-time electricity markets (e.g., balancing markets in Europe) where the memory of suppliers and consumers is taken into account. The latter is critical as it models the behaviour of market participants. For example, it is shown that market participants can use available market information to form price expectations and to exploit arbitrage opportunities[32]. Therefore, modelling such a behaviour is of utmost importance in current and future electricity markets.

3 | SOLUTIONS INVESTIGATION

Since the pencil of system (10) is regular there exist solutions for the system, see[21], and in addition $s\boldsymbol{E} - \boldsymbol{A}$ is also a regular pencil, see[26]. Because of the structure of $\boldsymbol{E}$ there exist invariants of the following type:

- μ finite eigenvalues of algebraic multiplicity p_i, $i = 1, ..., \mu, ..., 3$;
- an infinite eigenvalue of algebraic multiplicity q,

where $\sum_{i=1}^{\mu} p_i = p$, $p + q = 3$. There exist non-singular matrices $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{C}^{3\times 3}$ such that, see[33]:

$$\boldsymbol{PEQ} = \boldsymbol{I}_p \oplus \boldsymbol{H}_q, \quad \boldsymbol{PAQ} = \boldsymbol{J}_p \oplus \boldsymbol{I}_q \, , \tag{11}$$

where $\boldsymbol{J}_p \in \mathbb{C}^{p\times p}$ is a Jordan matrix, constructed by using the finite eigenvalues of the pencil and their algebraic multiplicities, $\boldsymbol{H}_q \in \mathbb{C}^{q\times q}$ is a nilpotent matrix with index q_*, constructed by using the infinite eigenvalue of the pencil and its algebraic multiplicity. We have the following cases:

1. The pencil of (10) to have all its eigenvalues finite. This is the most realistic case since it would mean that T_λ, T_{gi}, T_{di} are all non-zero. In this case let

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{1,n_1} \\ \boldsymbol{P}_{1,n_2} \\ \boldsymbol{P}_{1,n_3} \end{bmatrix}, \; \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{p,n_1} & \boldsymbol{Q}_{p,n_2} & \boldsymbol{Q}_{p,n_3} \end{bmatrix},$$

where $\boldsymbol{P}_{1,n_1} \in \mathbb{C}^{1\times 3}$, $\boldsymbol{P}_{1,n_2} \in \mathbb{C}^{1\times 3}$, $\boldsymbol{P}_{1,n_3} \in \mathbb{C}^{1\times 3}$, and $\boldsymbol{Q}_{p,n_1} \in \mathbb{C}^{3\times 1}$, $\boldsymbol{Q}_{p,n_2} \in \mathbb{C}^{3\times 1}$, $\boldsymbol{Q}_{p,n_3} \in \mathbb{C}^{3\times 1}$. Then (11) will take the form:

$$\boldsymbol{PEQ} = \boldsymbol{I}_p \, , \; \boldsymbol{PAQ} = \boldsymbol{J}_p \, .$$

We can write (10) in the form:

$$\begin{bmatrix} \frac{d^{n_1}}{dt^{n_1}} & 0 & 0 \\ 0 & \frac{d^{n_2}}{dt^{n_2}} & 0 \\ 0 & 0 & \frac{d^{n_3}}{dt^{n_3}} \end{bmatrix} \boldsymbol{Ex} = \boldsymbol{Ax} + \boldsymbol{\omega} \, .$$

By using the transformation $\boldsymbol{x} = \boldsymbol{Qz}$, then multiplying by $\boldsymbol{P}$ and using the above notation and (11) we get:

$$\boldsymbol{z}_{\hat{p}}^{(n_1)}(t) = \boldsymbol{J}_{\hat{p}} \boldsymbol{z}_{\hat{p}}(t) + \boldsymbol{P}_{1,n_1} \boldsymbol{\omega}(t) \, ;$$

$$\boldsymbol{z}_{\bar{p}}^{(n_2)}(t) = \boldsymbol{J}_{\bar{p}} \boldsymbol{z}_{\bar{p}}(t) + \boldsymbol{P}_{1,n_2} \boldsymbol{\omega}(t) \, ;$$

$$\boldsymbol{z}_{\tilde{p}}^{(n_3)}(t) = \boldsymbol{J}_{\tilde{p}} \boldsymbol{z}_{\tilde{p}}(t) + \boldsymbol{P}_{1,n_3} \boldsymbol{\omega}(t) \, ,$$

where

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{z}_{\hat{p}}(t) \\ \boldsymbol{z}_{\bar{p}}(t) \\ \boldsymbol{z}_{\tilde{p}}(t) \end{bmatrix}, \quad \boldsymbol{J}_p = \boldsymbol{J}_{\hat{p}} \oplus \boldsymbol{J}_{\bar{p}} \oplus \boldsymbol{J}_{\tilde{p}} \, .$$

We consider the first equation. By applying the Laplace transform $\mathcal{L}$ we get:

$$\mathcal{L}\{\boldsymbol{z}_{\hat{p}}^{(n_1)}(t)\} = \boldsymbol{J}_{\hat{p}}\mathcal{L}\{\boldsymbol{z}_{\hat{p}}(t)\} + \boldsymbol{P}_{1,n_1}\mathcal{L}\{\boldsymbol{\omega}(t)\}.$$

Let $\mathcal{L}\{\boldsymbol{z}_{\hat{p}}(t)\} = \boldsymbol{w}_{\hat{p}}(s)$. Then:

$$(s^{n_1}\boldsymbol{I}_{\hat{p}} - \boldsymbol{J}_{\hat{p}})\boldsymbol{w}_{\hat{p}}(s) = s^{n_1-1}\boldsymbol{C}_1 + \boldsymbol{P}_{1,n_1}\mathcal{L}\{\boldsymbol{\omega}(t)\}\ ,$$

or, equivalently,

$$\boldsymbol{w}_{\hat{p}}(s) = s^{\gamma-1}(s^{n_1}\boldsymbol{I}_{\hat{p}} - \boldsymbol{J}_{\hat{p}})^{-1}\boldsymbol{C}_1 + (s^{n_1}\boldsymbol{I}_{\hat{p}} - \boldsymbol{J}_{\hat{p}})^{-1}\boldsymbol{P}_{1,n_1}\mathcal{L}\{\boldsymbol{\omega}(t)\}\ .$$

By taking into account that $(s^{n_1}\boldsymbol{I}_{\hat{p}} - \boldsymbol{J}_{\hat{p}})^{-1} = \sum_{k=0}^{\infty} s^{-(k+1)n_1}\boldsymbol{J}_{\hat{p}}^k$ we have:

$$\boldsymbol{w}_{\hat{p}}(s) = \sum_{k=0}^{\infty} s^{-n_1k-1}\boldsymbol{J}_{\hat{p}}^k\boldsymbol{C}_1 + \sum_{k=0}^{\infty} s^{-(k+1)n_1}\boldsymbol{J}_{\hat{p}}^k\boldsymbol{P}_{1,n_1}\mathcal{L}\{\boldsymbol{\omega}(t)\}.$$

Then:

$$\boldsymbol{z}_{\hat{p}}(t) = \sum_{k=0}^{\infty} \frac{t^{n_1k}}{\Gamma(kn_1+1)}\boldsymbol{J}_{\hat{p}}^k\boldsymbol{C}_1 + \int_0^t \frac{(t-\tau)^{(k+1)n_1-1}}{\Gamma(kn_1+n_1)}\boldsymbol{J}_{\hat{p}}^k\boldsymbol{\omega}(\tau)d\tau\ .$$

To conclude, by similarly solving the other two equations we arrive at the general solution of (10) for this case:

$$\boldsymbol{x}(t) = \boldsymbol{Q}\boldsymbol{z}(t) = \sum_{i=1}^{3} \boldsymbol{Q}_{p,n_i}[\sum_{k=0}^{\infty} \frac{t^{n_ik}}{\Gamma(kn_i+1)}\boldsymbol{J}_i^k\boldsymbol{C}_i + \int_0^t \frac{(t-\tau)^{(k+1)n_i-1}}{\Gamma(kn_i+n_i)}\boldsymbol{J}_i^k\boldsymbol{\omega}(\tau)d\tau]\ . \tag{12}$$

Where $\boldsymbol{J}_1 = \boldsymbol{J}_{\hat{p}}$, $\boldsymbol{J}_2 = \boldsymbol{J}_{\bar{p}}$, $\boldsymbol{J}_3 = \boldsymbol{J}_{\tilde{p}}$.

2. The second case is the pencil of (10) to have an infinite eigenvalue. This means that at least one of the terms $T_\lambda, T_{gi}, T_{di}$ is zero or tends to zero. The number of terms that are zero is the algebraic multiplicity q of the infinite eigenvalue. Let T_{di} be the term that is zero but let T_λ, T_{gi} be strictly non-zero. Then the algebraic multiplicity q of the infinite eigenvalue is 1. Let:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{1,n_1} \\ \boldsymbol{P}_{1,n_2} \\ \boldsymbol{P}_{2,n_3} \end{bmatrix},\ \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{p,n_1} & \boldsymbol{Q}_{p,n_2} & \boldsymbol{Q}_{q,n_3} \end{bmatrix},$$

where $\boldsymbol{P}_{1,n_1} \in \mathbb{C}^{1\times3}$, $\boldsymbol{P}_{1,n_2} \in \mathbb{C}^{1\times3}$, $\boldsymbol{P}_{1,n_3} \in \mathbb{C}^{1\times3}$, and $\boldsymbol{Q}_{p,n_1} \in \mathbb{C}^{3\times1}$, $\boldsymbol{Q}_{p,n_2} \in \mathbb{C}^{3\times1}$, $\boldsymbol{Q}_{p,n_3} \in \mathbb{C}^{3\times1}$. The equations in (11) will take the form:

$$\boldsymbol{PEQ} = \boldsymbol{I}_{\hat{p}} \oplus \boldsymbol{I}_{\bar{p}} \oplus 0\ , \quad \boldsymbol{PAQ} = \boldsymbol{J}_{\hat{p}} \oplus \boldsymbol{J}_{\bar{p}} \oplus 1\ .$$

We can write (10) in the form:

$$\begin{bmatrix} \frac{d^{n_1}}{dt^{n_1}} & 0 & 0 \\ 0 & \frac{d^{n_2}}{dt^{n_2}} & 0 \\ 0 & 0 & \frac{d^{n_3}}{dt^{n_3}} \end{bmatrix}\boldsymbol{Ex} = \boldsymbol{Ax} + \boldsymbol{\omega}\ .$$

By using the transformation $\boldsymbol{x} = \boldsymbol{Qz}$, then multiplying by $\boldsymbol{P}$ and using (11) we get:

$$\boldsymbol{z}_{\hat{p}}^{(n_1)}(t) = \boldsymbol{J}_{\hat{p}}\boldsymbol{z}_{\hat{p}}(t) + \boldsymbol{P}_{1,n_1}\boldsymbol{\omega}(t)\ ;$$

$$\boldsymbol{z}_{\bar{p}}^{(n_2)}(t) = \boldsymbol{J}_{\bar{p}}\boldsymbol{z}_{\bar{p}}(t) + \boldsymbol{P}_{1,n_2}\boldsymbol{\omega}(t)\ ;$$

$$0 = \boldsymbol{z}_{\tilde{q}}(t) + \boldsymbol{P}_{2,n_3}\boldsymbol{\omega}(t)\ ,$$

where

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{z}_{\hat{p}}(t) \\ \boldsymbol{z}_{\bar{p}}(t) \\ \boldsymbol{z}_{\tilde{q}}(t) \end{bmatrix}.$$

The first two equations have same solutions as in case 1. The third equation has solution:

$$\boldsymbol{z}_{\tilde{q}}(t) = -\boldsymbol{P}_{2,n_3}\boldsymbol{\omega}(t)\ .$$

To conclude, by using $\boldsymbol{x}(t) = \boldsymbol{Qz}(t)$, we arrive at the general solution of (10) for this case:

$$\boldsymbol{x}(t) =$$

$$\sum_{i=1}^{2} \boldsymbol{Q}_{p,n_i}[\sum_{k=0}^{\infty} \frac{t^{n_i k}}{\Gamma(kn_i+1)} \boldsymbol{J}_i^k \boldsymbol{C}_i + \int_0^t \frac{(t-\tau)^{(k+1)n_i-1}}{\Gamma(kn_i+n_i)} \boldsymbol{J}_i^k \boldsymbol{\omega}(\tau) d\tau] - \boldsymbol{Q}_{q,n_3} \boldsymbol{P}_{2,n_3} \boldsymbol{\omega}(t) \ . \tag{13}$$

Where $\boldsymbol{J}_1 = \boldsymbol{J}_{\hat{p}}$, $\boldsymbol{J}_2 = \boldsymbol{J}_{\bar{p}}$.

We proved the following theorem:

Theorem 3.1. Using the spectrum of the pencil $s\boldsymbol{E} - \boldsymbol{A}$, the general solution of the fractional order system (10) is given by:

$$\boldsymbol{x}(t) = \sum_{i=1}^{3} f(T_i) \boldsymbol{Q}_{p,n_i}[\sum_{k=0}^{\infty} \frac{t^{n_i k}}{\Gamma(kn_i+1)} \boldsymbol{J}_i^k \boldsymbol{C}_i + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^{(k+1)n_i-1}}{\Gamma(kn_i+n_i)} \boldsymbol{J}_i^k \boldsymbol{\omega}(\tau) d\tau] - \sum_{i=1}^{3} g(T_i) \boldsymbol{Q}_{q,n_i} \boldsymbol{P}_{2,n_i} \boldsymbol{\omega}(t) \ . \tag{14}$$

Where $T_1 = T_\lambda$, $T_2 = T_{gi}$, $T_3 = T_{di}$. The matrices $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{J}_3$ are Jordan matrices defined in (11), (12), (13), and constructed by the finite eigenvalues of the pencil $s\boldsymbol{E} - \boldsymbol{A}$, and their algebraic multiplicity, while $\boldsymbol{Q}_{p,n_i} \in \mathbb{C}^{3\times p}$ are the matrices constructed by the linear independent eigenvectors related to the finite eigenvalues of the pencil. $\boldsymbol{C}_i \in \mathbb{C}^{p\times 1}$ are constant vectors. The matrices $\boldsymbol{Q}_{q,n_i}$, $\boldsymbol{P}_{2,n_i}$ are matrices with left and right eigenvectors of the infinite eigenvalue. Finally

$$f(T_i) = \left\{ \begin{array}{l} 1, \ T_i \neq 0 \\ 0, \ T_i = 0 \end{array} \right\}, \quad g(T_i) = 1 - f(T_i).$$

4 | EXAMPLES

As a first example we assume system (10) with $T_\lambda = H_q = c_{g_i} = 1$, $T_{gi} = -c_{d_i} = \frac{1}{2}$, $T_{di} = \frac{1}{6}$. The pencil $s\boldsymbol{E} - \boldsymbol{A}$ has three finite eigenvalues $s_1 = -1$, $s_2 = -2$, $s_3 = -3$ with eigenvectors

$$\boldsymbol{Q}_{p,n_1} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{Q}_{p,n_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{Q}_{p,n_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

respectively. Hence the solution of the system is equal to:

$$\boldsymbol{x}(t) = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} [\sum_{k=0}^{\infty} \frac{t^{n_1 k}}{\Gamma(kn_1+1)} (-1)^k C_1 + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^{(k+1)n_1-1}}{\Gamma(kn_1+n_1)} (-1)^k \omega(\tau) d\tau] + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [\sum_{k=0}^{\infty} \frac{t^{n_2 k}}{\Gamma(kn_2+1)} (-2)^k C_2 + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^{(k+1)n_2-1}}{\Gamma(kn_2+n_2)} (-2)^k \omega(\tau) d\tau] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [\sum_{k=0}^{\infty} \frac{t^{n_3 k}}{\Gamma(kn_3+1)} (-3)^k C_3 + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^{(k+1)n_3-1}}{\Gamma(kn_3+n_3)} (-3)^k \omega(\tau) d\tau] \ .$$

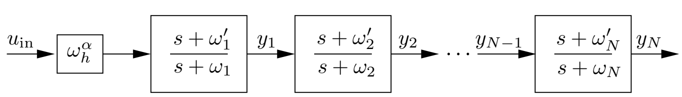

FIGURE 3 *Oustaloup's recursive approximation block diagram.*

In order to implement or simulate in practice the proposed fractional market model (7)-(9), one needs to approximate the fractional dynamics. In our second example we will keep the order of equation (7) fractional $n_1 = \alpha$, $0 < \alpha < 1$, but will assume that $n_2 = n_3 = 1$. We will use the Oustaloup's Recursive Approximation (ORA) method to approximate the fractional electricity price dynamics. The generalized ORA of a fractional derivative of order α is defined as[34]:

$$s^\alpha \approx \omega_h^\alpha \prod_{k=1}^{N} \frac{s + \omega'_k}{s + \omega_k}, \tag{15}$$

where $\omega'_k = \omega_b \omega_v^{(2k-1-\alpha)/N}$, $\omega_k = \omega_b \omega_v^{(2k-1+\alpha)/N}$, $\omega_v = \sqrt{\omega_h/\omega_b}$. In the above expressions, $[\omega_b, \omega_h]$ is the frequency range for which the approximation is designed to be valid; N is the order of the polynomial approximation. The term "generalized" means that, in (15), N can be either even or odd[34], while the term "recursive" implies that the values of ω'_k, ω_k result from a set of recursive equations. The block diagram of ORA is shown in Fig. 3 . Further details on the ORA method and its accuracy can be found in[27] and references therein.

In addition to the simulations of this example we provide a comparison between the conventional integer-order MAGC (I-MAGC) (2)-(4), $n_1 = n_2 = n_3 = 1$, and the fractional-order MAGC (F-MAGC) (7)-(9) with $n_1 = \alpha$, $n_2 = n_3 = 1$. The objective is to evaluate the impact of these models on the behavior of market participants, e.g. generator schedules, and on the overall dynamic response of the power system.

The case study is based on a modified version of the well-known WSCC 9-bus test system, whose details are provided in[35]. All simulations below are performed using the Python-based power system analysis software tool Dome[36]. Note that Dome allows simulating larger networks (e.g., thousand of buses). In this case, the main difference will be an increase in the computational burden of the simulation.

Some long-term power system dynamics, e.g. the dynamics of the AGC, evolve with a timescale similar to today's short-term market dynamics[2]. For this reason, it is important to understand how the frequency with which the market price is updated impacts on the decision-making process of market participants and on power system dynamics. In the continuous market models considered in this paper, the information on how often the price is updated is contained in the value of the gain K_E in (2).

Figure 4 shows that the value of K_E has a negligible impact on the overall dynamic of the sysytem, i.e. the frequency nadir is the same in all cases. This was expected as the MAGC is slow with respect to the primary frequency control. Figure 5 , on the other hand, shows that the schedule of generator active power are by the value of K_E. Specifically, the faster the price updates, i.e. the higher K_E, the faster the generator response and consequently the higher the generator schedules. These results indicate that how often the market updates the price (which in this continuous model is modelled by means of K_E) impacts the schedule of the suppliers or generators.

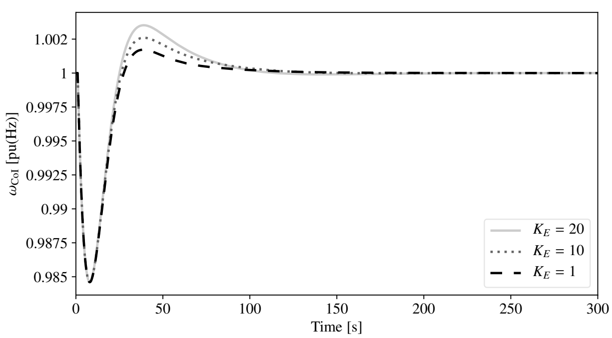

FIGURE 4 Trajectories of the frequency of the CoI.

The trajectories of the AGC set-point ΔP_1 of generator 1 are shown in Fig. 6 . Higher gain values – and hence faster price updates – lead to faster AGC response and lower AGC set-points. This has to be expected as the AGC has to compensate the difference in the market schedules since at the end the total power output of the generator has to be the same. These results imply that, depending on the market design and rewards of the ancillary services, generators may prefer to compensate power unbalances through the short-term market or through the secondary frequency control.

Figure 7 shows the trajectories of ω_{CoI} for both models. It is interesting to observe that both the I-MAGC and F-MAGC lead to the same frequency nadir and very similar frequency overshoots. The memory of market participants, thus, does not have a relevant impact on the overall power system dynamics. These results are consistent with those shown in Fig. 4 .

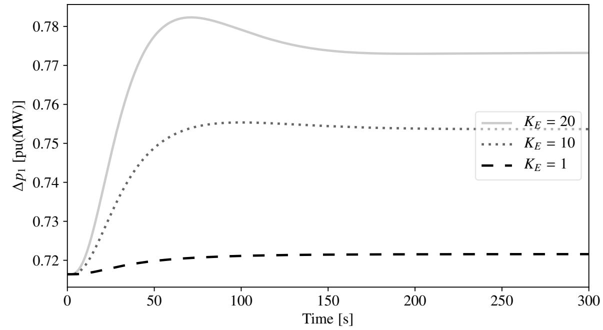

FIGURE 5 *Trajectories of the MAGC active power schedules of generator 1.*

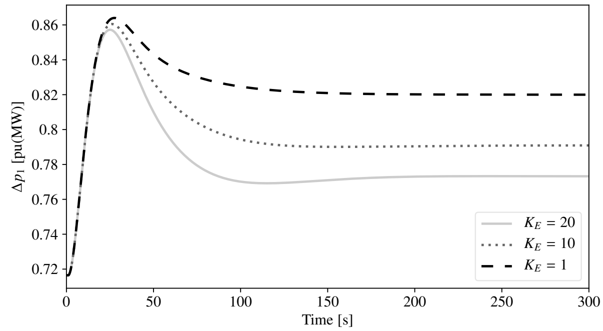

FIGURE 6 *Trajectories of the AGC active power set-point of generator 1.*

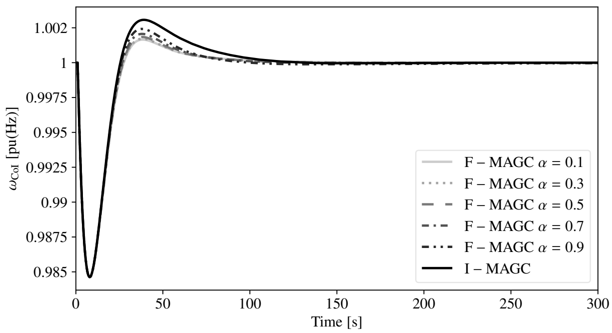

FIGURE 7 *Comparison of the trajectories of the frequency of the CoI as obtained with the I-MAGC and F-MAGC.*

Next we compare the impact that different values of α have on the behavior of the generators. Figure 8 shows that the F-MAGC leads to different (in this case, lower) market schedules as compared to that of the I-MAGC. This result suggests that the F-MAGC is less prone to the price changes. In other words, taking into account the memory of market participants makes them more conservative. This conclusion is supported by Fig. 9 . This figure shows that the AGC set-point for the fractional market is less prone to changes compared to the conventional market. Furthermore, the higher the fractional order α, the faster the generator response, and consequently the higher the generator market schedules.

Finally, we compare the impact on the performance of I-MAGC and F-MAGC of a 10% sudden load decrease occurring at $t = 1$ s.

Figure 10 shows that the F-MAGC is again less prone to price changes compared to the I-MAGC. For the considered contingency, such a behaviour leads the market to schedule higher generator powers.

Figure 11 shows the AGC power output and indicates that the I-MAGC case responds faster than the F-MAGC to the contingency. This result is consistent with that obtained in the previous section, i.e. the memory effect makes the market participants

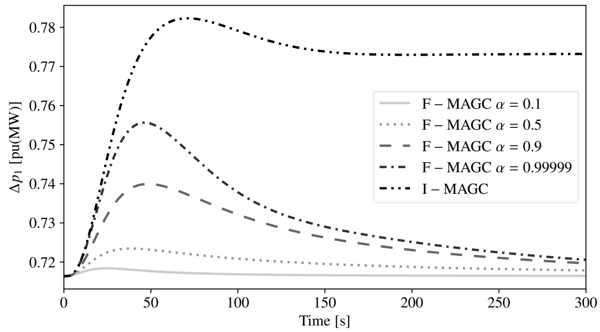

FIGURE 8 *Trajectories of the MAGC active power schedules of generator 1.*

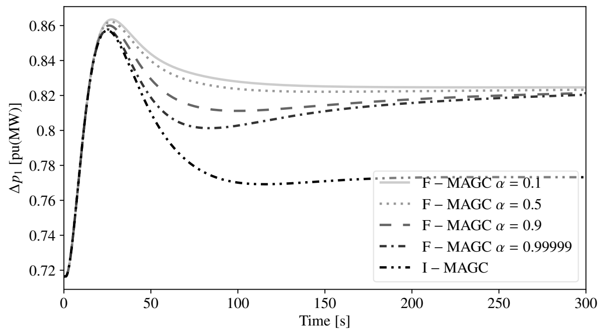

FIGURE 9 *Trajectories of the AGC active power set-point of generator 1.*

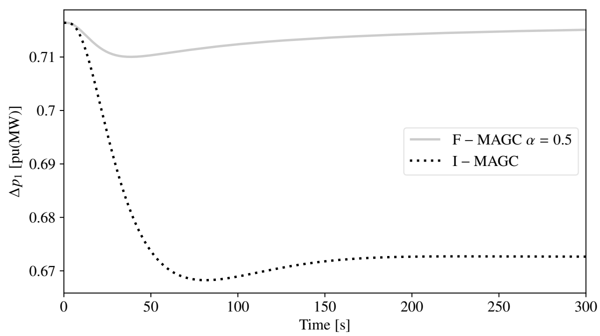

FIGURE 10 *Trajectories of the MAGC active power schedules of generator 1.*

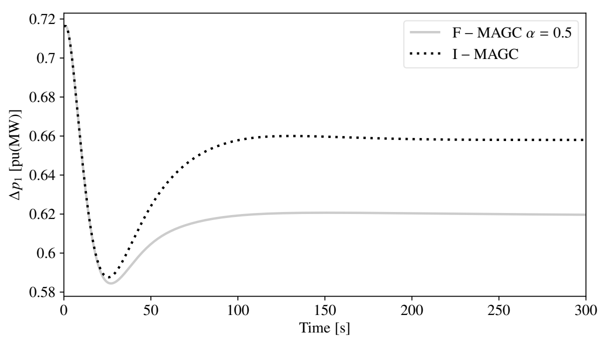

FIGURE 11 *Trajectories of the AGC active power set-point of generator 1.*

less sensitive to changes in the operating point of the grid. This conservativeness, however, has to be compensated, at least in the short term, by the secondary frequency regulation.

5 | CONCLUDING REMARKS

In this article we constructed a dynamical model for electricity markets based on differential equations of fractional order. We studied then its solutions and provided both analytical and numerical examples including a comparison to a model of integer order differential equations. We can conclude that beside electricity markets, the proposed method can be used in the studies of other similar models where the memory effect appears. In fact, this type of operators are very useful tools for time scale analysis with applications in macroeconomic problems[37,38,39], electrical power systems[40] and energy storage models[41].

As a further extension of this article, we aim to study the stability of the equilibriums of this system, perturbation methods and construct optimization techniques in order to obtain optimal solutions for the case of existence but not uniqueness of solutions for this system. Finally, we aim to consider and introduce additional fractional operators such as forward fractional operators which, unlike the backward operator used in this article, emphasize on future predictions. For all this there is already some research in progress.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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