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Singular dual systems of fractional order differential equations

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Summary

We consider both primal and dual formulations of singular autonomous systems of three different types of fractional order differential equations. We present a comprehensive study which proves that by using the spectrum of a linear pencil, a polynomial matrix of first order, and not the fractional order pencil of the prime system we will receive information for all properties for both the prime, and its dual system. In addition, by using this spectrum, the solutions for all systems can be obtained by using formulas without additional computational cost. Finally, we provide examples including a computational analysis in Modelica.

KEYWORDS:
singular, dual, differential, fractional, modelica

1 | INTRODUCTION

Singular linear systems of differential & difference equations appear in control theory[4],[19],[25], circuit theory[22],[5] and in the modeling (dynamics) of electrical power systems[11],[24],[32].

In the last decade, many authors have studied problems of fractional differential/difference equations and have derived interesting results on different types of problems for given initial or boundary conditions, see[2],[9],[10],[12],[13],[14],[20],[30],[34]. Focus has also been given on the mathematical modelling of many phenomena by using fractional operators. The theory of fractional differential equations (FDEs) is a promising tool for applications in neural networks[29], in physics[8],[31], biology[17], and control theory, see[6],[33].

Despite several studies, there are still parts missing for a complete and coherent theory of systems of FDEs in order to use this type of systems as a tool in mathematical modeling in a similar way to the classical case. In addition, generalized FDEs and cases such as singularities of certain systems of FDEs have been mostly avoided in the framework of fractional calculus. Hence, explicit and easily testable methods are required in order to solve generalized systems of FDEs, so that applied researchers can redesign their models using fractional operators where this is appropriate.

In this article we consider both primal and dual formulations of singular systems of FDEs. We use the Caputo fractional derivative and two recently defined alternative versions of this derivative, the Caputo–Fabrizio, and the Atangana–Baleanu fractional operator.

We consider the following system of fractional differential equations (FDEs):

$$\boldsymbol{E}\,{}_0D_t^a\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t), \tag{1}$$

and its dual version

$$\boldsymbol{A}\,{}_0D_t^a\tilde{\boldsymbol{x}}(t) = \boldsymbol{E}\tilde{\boldsymbol{x}}(t). \tag{2}$$

Where $\boldsymbol{x} : [0, +\infty] \to \mathbb{C}^{m\times 1}$, $\boldsymbol{E}, \boldsymbol{A} \in \mathbb{C}^{m\times m}$, with $\boldsymbol{E}$ singular (det$\boldsymbol{E}$=0). We use three different definitions for the fractional-a order operator ${}_0D_t^a$:

Definition 1.1. Let $\boldsymbol{x} : [0, +\infty) \rightarrow \mathbb{R}^{m\times 1}$, $t \rightarrow \boldsymbol{x}$, denote a column of differentiable functions. Then:

- The Caputo (C) fractional derivative of order $0 < a < 1$, is defined by, see[2,18]:

$$ {}_0D_t^a\boldsymbol{x}(t) := {}_0^C D_t^a\boldsymbol{x}(t) = \frac{1}{\Gamma(1-a)}\int_0^t \left[(t-\tau)^{-a}\boldsymbol{x}'(\tau)\right] d\tau \, . \tag{3}$$

- The Caputo–Fabrizio (CF) fractional derivative of order $0 \le a \le 1$, is defined by, see[3]:

$$ {}_0D_t^a\boldsymbol{x}(t) := {}^{CF}_{\ 0}D_t^a\boldsymbol{x}(t) = \frac{1}{1-a}\int_0^t \left[e^{-\frac{a}{1-a}(t-\tau)}\boldsymbol{x}'(\tau)\right] d\tau. \tag{4}$$

- The Atangana–Baleanu in Caputo sense (ABC) fractional derivative of order $0 \le a \le 1$, is defined by, see[1]:

$$ {}_0D_t^a\boldsymbol{x}(t) := {}^{ABC}_{\ \ \ 0}D_t^a\boldsymbol{x}(t) = \frac{B(a)}{1-a}\int_0^t E_a\left[-a\frac{(t-\tau)^a}{1-a}\right]\boldsymbol{x}'(\tau)d\tau. \tag{5}$$

  Where $E_a(\omega) = \sum_{k=0}^{\infty}\frac{\omega^k}{\Gamma(1+ak)}$, see[2],[18]. $B(a)$ is a normalized function with $B(0) = B(1) = 1$.

In terms of applications, the Caputo fractional derivative (C) is used widely in the literature but as stated in[1],[3], the changes made in the kernel of the Caputo fractional derivative, in order to provide these new alternative versions, can capture more efficiently several phenomena. A comparison of this three fractional operators can be found when modelling the heat transfer at nanoscale, i.e. nanometers & picoseconds, see[8].

Some very recent results on applications of singular systems of FDEs include the construction of a model of fractional order controllers for electrical power system applications, see[33], while by using the properties of the fractional order derivative, in terms of memory, a new model of singular FDEs is constructed to represent more efficiently electricity markets, see[21]. In both these models duality analysis can be a fundamental tool for the small-signal stability analysis of the power systems as it can provide insight and knowledge on the solutions, the invariants and the linear programming used, see[7],[23].

In these applications the question that arises is which one of the three fractional operators, defined in (3), (4), and (5), is the most appropriate to use. The answer to this question defers for each application. If we consider the first application mentioned above, the fractional order depends on the reason of why the fractional controller is used. If for example the problem is a stability issue then the operator which can be stabilise more quickly or efficiently and with less computational cost the system is the one which is more efficient. In this case and by using the formulas provided in Theorem 3.1 in Section 3, even optimization methods might be required to obtain the optimal solution. However, it is worth mentioning that to the best of our knowledge, see also[32], the (CF), (AB) have never been used in fractional control theory. In addition, the duality results obtained in this article will help researchers who are using this type of controllers to determine eigenvalues with very large real parts. All this is a great prospect for future research.

Using the Laplace transform $\mathcal{L}$ into (1), see[2],[9],[18], we have:

$$\mathcal{L}\{\boldsymbol{E}\,{}_0D_t^a\boldsymbol{x}(t)\} = \mathcal{L}\{\boldsymbol{A}\boldsymbol{x}(t)\},$$

or, equivalently,

$$z\boldsymbol{E}\boldsymbol{w}(s) - w\boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{w}(s),$$

or, equivalently,

$$(z\boldsymbol{E} - \boldsymbol{A})\boldsymbol{w}(s) = w\boldsymbol{x}(0).$$

Where $\mathcal{L}\{\boldsymbol{x}(t)\} = \boldsymbol{w}(s)$, and if ${}_0D_t^a$ is the fractional derivative by definition of

(i) (C) then $z := z(s) = s^a$, $w = s^{a-1}$;

(ii) (CF) then $z := z(s) = \frac{s}{s+a(1-s)}$, $w = \frac{1}{s+a(1-s)}$;

(iii) (ABC) then $z := z(s) = \frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}$, $w = \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}$.

The pencil of (1) will then be $zE - A$. For each different definition of the fractional operators, (C), (CF), (ABC), z is defined differently. It can be observed that it is complicated to use the non-linear pencil $zE - A$ to study system (1). Instead of using this pencil we will use the pencil $sE - A$, i.e. the pencil of system $Ex'(t) = Ax(t)$, to study (1) for the three different definitions of fractional derivatives, and its respectively dual systems (2). In addition, we will prove that if the solution of this first order system is known, then the solutions for both prime and dual systems of FDEs can be obtained without additional computational cost. To sum up, the results in this article prove that despite having a fractional order system, we can use the spectrum of a pencil of first order to have information for all these systems, both prime & dual, independently the definition of fractional derivative used and without any additional computational cost.

2 | PRELIMINARIES AND USEFUL TOOLS

The pencil $sE - A$ can have eigenvalues: $\lambda_0 = 0$, $\lambda_i \neq 0$ with $i = 1, 2, ..., \nu - 1$, $\lambda_\infty \to \infty$, with algebraic multiplicities p_0, p_i, q respectively, and $p_0 + \sum_{i=1}^{\nu} p_i = p$, $p + q = m$. Let $B_{n_1} \in \mathbb{C}^{n_1 \times n_1}$,$B_{n_2} \in \mathbb{C}^{n_2 \times n_2}$, …, $B_{n_r} \in \mathbb{C}^{n_r \times n_r}$. Then, the block diagonal matrix blockdiag $\left(B_{n_1}, B_{n_2}, \dots, B_{n_r}\right)$ will be denoted via the direct sum $B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r}$. There exist invertible matrices $P, Q \in \mathbb{C}^{m \times m}$ such that

$$PEQ = I_{p_0} \oplus I_p \oplus H_q, \quad PAQ = J_{p_0} \oplus J_p \oplus I_q. \tag{6}$$

Where $J_{p_0} \in \mathbb{C}^{p_0 \times p_0}$, $J_p \in \mathbb{C}^{p \times p}$, $H_q \in \mathbb{C}^{q \times q}$ are Jordan matrices related to the zero eigenvalue, the non-zero finite eigenvalues, infinite eigenvalue respectively, see chapter 12 in[16], while P, Q contain the left, right eigenvectors of the eigenvalues respectively. Let

$$P = \begin{bmatrix} P_{p_0} \\ P_p \\ P_q \end{bmatrix}, \quad Q = \left[\, Q_{p_0} \; Q_p \; Q_q \,\right],$$

where $P_{p_0} \in \mathbb{C}^{p_0 \times m}$, $P_p \in \mathbb{C}^{p \times m}$, $P_q \in \mathbb{C}^{q \times m}$, and, $Q_{p_0} \in \mathbb{C}^{m \times p_0}$, $Q_p \in \mathbb{C}^{m \times p}$, $Q_q \in \mathbb{C}^{m \times q}$ are matrices with rows the left, and columns the right respectively eigenvectors of the zero, non-zero, infinite eigenvalues.

The pencil $\tilde{s}A - E$ can then have eigenvalues: $\tilde{\lambda}_0 = lim_{\lambda_\infty \to \infty} \frac{1}{\lambda_\infty} = 0$, $\tilde{\lambda}_i = \frac{1}{\lambda_i} \neq 0$ with $i = 1, 2, ..., \nu$, and $\tilde{\lambda}_\infty = \frac{1}{\lambda_0} \to \infty$. Then if $\tilde{\nu}$ is the number of the finite eigenvalues (zero & non-zero) of $\tilde{s}A - E$:

- If $0, \infty$ are eigenvalues of $sE - A$ then $\tilde{\nu} = \nu$;
- If 0 is an eigenvalue of $sE - A$ but ∞ is not then $\tilde{\nu} = \nu - 1$;
- If 0 is not an eigenvalue of $sE - A$ but ∞ is then $\tilde{\nu} = \nu + 1$.

There exist invertible matrices $\tilde{P}, \tilde{Q} \in \mathbb{C}^{m \times m}$, see chapter 12 in[16], such that:

$$\tilde{P}A\tilde{Q} := \tilde{A}_w = I_{\tilde{p}} \oplus \tilde{H}_{\tilde{q}}, \quad \tilde{P}E\tilde{Q} := \tilde{E}_w = \tilde{J}_{\tilde{p}} \oplus I_{\tilde{q}}. \tag{7}$$

Let

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{\tilde{p}} \\ \tilde{P}_{\tilde{q}} \end{bmatrix} \quad \tilde{Q} = \left[\, \tilde{Q}_{\tilde{p}} \; \tilde{Q}_{\tilde{q}} \,\right],$$

with $\tilde{p}$ bethe sum of all algebraic multiplicities of the infinite eigenvalues, and $\tilde{q} = p_0$ algebraic multiplicity of the infinite eigenvalue. Where $\tilde{P}_{\tilde{p}} \in \mathbb{C}^{\tilde{p} \times m}$, $\tilde{P}_{\tilde{q}} \in \mathbb{C}^{\tilde{q} \times m}$, and $\tilde{Q}_{\tilde{p}} \in \mathbb{C}^{m \times \tilde{p}}$, $\tilde{Q}_{\tilde{q}} \in \mathbb{C}^{m \times \tilde{q}}$. Where $\tilde{P}_{\tilde{p}}, \tilde{P}_{\tilde{q}}$ are matrices with rows, and columns of the left, and right eigenvectors of the finite eigenvalues, and the infinite eigenvalue respectively.

Theorem 2.1. We consider the pencils $sE - A$, $\tilde{s}A - E$ with $E, A \in \mathbb{C}^{m \times m}$, and E singular. Let P, Q, and $\tilde{P}, \tilde{Q}$ be the matrices defined in (6), and (7) respectively. Then:

$$\tilde{P}_{\tilde{p}} = \begin{bmatrix} P_p \\ P_q \end{bmatrix}, \quad \tilde{Q}_{\tilde{p}} = \left[\, Q_p \; Q_q \,\right]. \tag{8}$$

Proof. We consider the system $Ex'(t) = Ax(t)$, and substitute the transformation $x(t) = Qz(t)$. By multiplying by P we obtain

$$PEQz'(t) = PAQz(t).$$

Let

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{z}_{p_0}(t) \\ \boldsymbol{z}_p(t) \\ \boldsymbol{z}_q(t) \end{bmatrix},$$

with $\boldsymbol{z}_{p_0}(t) \in \mathbb{C}^{p_0 \times 1}$, $\boldsymbol{z}_p(t) \in \mathbb{C}^{p \times 1}$, $\boldsymbol{z}_q(t) \in \mathbb{C}^{q \times 1}$. Then by using the form of $\boldsymbol{Q}$ in (6), we arrive at three subsystems:

$$\boldsymbol{z}'_{p_0}(t) = \boldsymbol{J}_{p_0}\boldsymbol{z}_{p_0}(t);$$

$$\boldsymbol{z}'_p(t) = \boldsymbol{J}_p\boldsymbol{z}_p(t);$$

$$\boldsymbol{H}_q\boldsymbol{z}'_q(t) = \boldsymbol{z}_q(t).$$

The first two subsystems have solutions:

$$\boldsymbol{z}_{p_0}(t) = e^{\boldsymbol{J}_{p_0}t}\boldsymbol{z}_{p_0}(0), \quad \boldsymbol{z}_p(t) = e^{\boldsymbol{J}_p t}\boldsymbol{z}_p(0).$$

If with $\boldsymbol{0}_{ij}$ we indicate the zero matrix of i rows, j columns respectively, and with q_* the index of the nilpotent matrix $\boldsymbol{H}_q$ such that $\boldsymbol{H}_q^{q_*} = \boldsymbol{0}_{q,q}$, then if we take the third subsystem and repeatedly multiply by $\boldsymbol{H}_q$:

$$\begin{array}{c} \boldsymbol{H}_q\boldsymbol{z}'_q(t) = \boldsymbol{z}_q(t) \\ \boldsymbol{H}_q^2\boldsymbol{z}''_q(t) = \boldsymbol{H}_q\boldsymbol{z}'_q(t) \\ \boldsymbol{H}_q^3\boldsymbol{z}'''_q(t) = \boldsymbol{H}_q^2\boldsymbol{z}''_q(t) \\ \boldsymbol{H}_q^4\boldsymbol{z}_q^{(4)}(t) = \boldsymbol{H}_q^3\boldsymbol{z}'''_q(t) \\ \vdots \\ \boldsymbol{H}_q^{q_*-1}\boldsymbol{z}_q^{(q_*-1)}(t) = \boldsymbol{H}_q^{q_*-2}\boldsymbol{z}_q^{(q_*-2)}(t) \\ \boldsymbol{H}_q^{q_*}\boldsymbol{z}_q^{(q_*)}(t) = \boldsymbol{H}_q^{q_*-1}\boldsymbol{z}_q^{(q_*-1)}(t) \end{array}.$$

And the sum of it gives:

$$\Big(\sum_{i=1}^{q_*-1}\boldsymbol{H}_q^i\boldsymbol{z}_q^{(i)}(t)\Big) + \boldsymbol{H}_q^{q_*}\boldsymbol{z}_q^{(q_*)}(t) = \Big(\sum_{i=1}^{q_*-1}\boldsymbol{H}_q^i\boldsymbol{z}_q^{(i)}(t)\Big) + \boldsymbol{z}_q(t),$$

or, equivalently, by taking into account that $\boldsymbol{H}_q^{q_*} = \boldsymbol{0}_{q,q}$, at the solution:

$$\boldsymbol{z}_q(t) = \boldsymbol{0}_{q,1}.$$

Consequently, we obtain:

$$\boldsymbol{x}(t) = \boldsymbol{Q}\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix} \begin{bmatrix} e^{\boldsymbol{J}_{p_0}t}\boldsymbol{z}_{p_0}(0) \\ e^{\boldsymbol{J}_p t}\boldsymbol{z}_p(0) \\ \boldsymbol{0}_{q,1} \end{bmatrix},$$

or, equivalently,

$$\boldsymbol{x}(t) = \boldsymbol{Q}_{p_0}e^{\boldsymbol{J}_{p_0}t}\boldsymbol{z}_{p_0}(0) + \boldsymbol{Q}_p e^{\boldsymbol{J}_p t}\boldsymbol{z}_p(0),$$

or, equivalently,

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p \end{bmatrix} e^{\boldsymbol{J}_{p_0+p}t}\boldsymbol{z}_{p_0+p}(0),$$

where $e^{\boldsymbol{J}_{p_0+p}t} = e^{\boldsymbol{J}_{p_0}t} \oplus e^{\boldsymbol{J}_p t}$, and $\boldsymbol{z}_{p_0+p}(0) = \begin{bmatrix} \boldsymbol{z}_{p_0}(0) \\ \boldsymbol{z}_p(0) \end{bmatrix}$ is a constant vector. This means that $\begin{bmatrix} \boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p \end{bmatrix}$ is the matrix that contains the $p_0 + p$ linear independent eigenvectors of the finite eigenvalues of $s\boldsymbol{E} - \boldsymbol{A}$. Let us now consider the system $\boldsymbol{A}\tilde{\boldsymbol{x}}' = \boldsymbol{E}\tilde{\boldsymbol{x}}$. We then apply the transformation $\tilde{\boldsymbol{x}}(t) = \boldsymbol{Q}\tilde{\boldsymbol{z}}(t)$, multiply by $\boldsymbol{P}$, and arrive at:

$$\boldsymbol{P}\boldsymbol{A}\boldsymbol{Q}\tilde{\boldsymbol{z}}'(t) = \boldsymbol{P}\boldsymbol{E}\boldsymbol{Q}\tilde{\boldsymbol{z}}(t),$$

or, equivalently, by using (6):

$$[\boldsymbol{J}_{p_0} \oplus \boldsymbol{J}_p \oplus \boldsymbol{I}_q]\tilde{\boldsymbol{z}}'(t) = [\boldsymbol{I}_{p_0} \oplus \boldsymbol{I}_p \oplus \boldsymbol{H}_q]\tilde{\boldsymbol{z}}(t),$$

whereby setting

$$\tilde{\boldsymbol{z}}(t) = \begin{bmatrix} \tilde{\boldsymbol{z}}_{p_0}(t) \\ \tilde{\boldsymbol{z}}_p(t) \\ \tilde{\boldsymbol{z}}_q(t) \end{bmatrix},$$

where $\tilde{\boldsymbol{z}}_{p_0}(t) \in \mathbb{C}^{p_0\times 1}$, $\tilde{\boldsymbol{z}}_p(t) \in \mathbb{C}^{p\times 1}$, $\tilde{\boldsymbol{z}}_q(t) \in \mathbb{C}^{q\times 1}$, we obtain:

$$\boldsymbol{J}_{p_0}\tilde{\boldsymbol{z}}'_{p_0}(t) = \boldsymbol{I}_{p_0}\tilde{\boldsymbol{z}}_{p_0}(t);$$

$$\boldsymbol{J}_{p}\tilde{\boldsymbol{z}}'_{p}(t) = \boldsymbol{I}_{p}\tilde{\boldsymbol{z}}_{p}(t);$$

$$\boldsymbol{I}_{q}\tilde{\boldsymbol{z}}'_{q}(t) = \boldsymbol{H}_{q}\tilde{\boldsymbol{z}}_{q}(t).$$

The matrix $\boldsymbol{J}_{p_0}$ is nilpotent because there are only zeros in its diagonal. Furthermore, the matrices $\boldsymbol{J}_p$, $\boldsymbol{I}_q$ are both invertible since they are either upper triangular matrices or diagonal with non-zero elements in its main diagonal. Since the first subsystem is similar to $\boldsymbol{H}_q\boldsymbol{z}'_q(t) = \boldsymbol{z}_q(t)$ we have:

$$\tilde{\boldsymbol{z}}_{p_0}(t) = \boldsymbol{0}_{p_0,1}.$$

For the other two:

$$\tilde{\boldsymbol{z}}_p(t) = e^{\tilde{\boldsymbol{J}}_p t}\tilde{\boldsymbol{z}}_p(0), \quad \text{and} \quad \tilde{\boldsymbol{z}}_q(t) = e^{\tilde{\boldsymbol{J}}_q t}\tilde{\boldsymbol{z}}_q(0),$$

Where

$$\tilde{\boldsymbol{J}}_p = (\boldsymbol{J}_p)^{-1}, \quad \tilde{\boldsymbol{J}}_q = \boldsymbol{H}_q.$$

Consequently by using $\boldsymbol{Q}$ as defined in (8) we obtain:

$$\tilde{\boldsymbol{x}}(t) = Q\tilde{z}(t) = \begin{bmatrix} \boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix} \begin{bmatrix} \boldsymbol{0}_{p_0,1} \\ e^{\tilde{\boldsymbol{J}}_p t}\tilde{\boldsymbol{z}}_p(0) \\ e^{\tilde{\boldsymbol{J}}_q t}\tilde{\boldsymbol{z}}_q(0) \end{bmatrix},$$

or, equivalently,

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{Q}_p e^{\tilde{\boldsymbol{J}}_p t}\tilde{\boldsymbol{z}}_p(0) + \boldsymbol{Q}_q e^{\tilde{\boldsymbol{J}}_q t}\tilde{\boldsymbol{z}}_q(0),$$

or, equivalently,

$$\tilde{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix} e^{\boldsymbol{J}_{p+q} t}\tilde{\boldsymbol{z}}_{p+q}(0),$$

where $e^{\boldsymbol{J}_{p+q} t} = e^{\tilde{\boldsymbol{J}}_p t} \oplus e^{\tilde{\boldsymbol{J}}_q t}$, $\tilde{\boldsymbol{z}}_{p+q}(0) = \begin{bmatrix} \tilde{\boldsymbol{z}}_p(0) \\ \tilde{\boldsymbol{z}}_q(0) \end{bmatrix}$. Hence $\begin{bmatrix} \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix}$ has as columns the linear independent eigenvectors of the finite eigenvalues of $\tilde{s}\boldsymbol{A} - \boldsymbol{E}$. Thus:

$$\tilde{\boldsymbol{Q}}_{\tilde{p}} = \begin{bmatrix} \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix}.$$

Let us now consider the system

$$\boldsymbol{A}^T\tilde{\boldsymbol{x}}'^T = \boldsymbol{E}^T\tilde{\boldsymbol{x}}^T,$$

or, equivalently,

$$\tilde{\boldsymbol{x}}'\boldsymbol{A} = \tilde{\boldsymbol{x}}\boldsymbol{E}.$$

Where $\tilde{\boldsymbol{x}} \in \mathbb{C}^{1\times m}$. We apply the transformation

$$\tilde{\boldsymbol{x}}(t) = \tilde{z}(t)\boldsymbol{P}$$

into the above system, and multiply by $\boldsymbol{Q}$:

$$\tilde{z}'(t)\boldsymbol{PAQ} = \tilde{z}(t)\boldsymbol{PEQ},$$

or, equivalently,

$$\tilde{z}'(t)[\boldsymbol{J}_{p_0} \oplus \boldsymbol{J}_p \oplus \boldsymbol{I}_q] = \tilde{z}(t)[\boldsymbol{I}_{p_0} \oplus \boldsymbol{I}_p \oplus \boldsymbol{H}_q],$$

whereby setting

$$\tilde{z}(t) = \begin{bmatrix} \tilde{\boldsymbol{z}}_{p_0}(t) & \tilde{\boldsymbol{z}}_p(t) & \tilde{\boldsymbol{z}}_q(t) \end{bmatrix},$$

with $\tilde{\boldsymbol{z}}_{p_0}(t) \in \mathbb{C}^{1\times p_0}$, $\tilde{\boldsymbol{z}}_p(t) \in \mathbb{C}^{1\times p}$, $\tilde{\boldsymbol{z}}_q(t) \in \mathbb{C}^{1\times q}$, and using the above written notations we arrive at three subsystems:

$$\tilde{\boldsymbol{z}}'_{p_0}(t)\boldsymbol{J}_{p_0} = \tilde{\boldsymbol{z}}_{p_0}(t)\boldsymbol{I}_{p_0};$$

$$\tilde{\boldsymbol{z}}'_{p}(t)\boldsymbol{J}_{p} = \tilde{\boldsymbol{z}}_{p}(t)\boldsymbol{I}_{p};$$

$$\tilde{\boldsymbol{z}}'_{q}(t)\boldsymbol{I}_{q} = \tilde{\boldsymbol{z}}_{q}(t)\boldsymbol{H}_{q}.$$

As already written, the matrix $\boldsymbol{J}_{p_0}$ has only zeros in its diagonal. Furthermore, the matrices $\boldsymbol{J}_p$, $\boldsymbol{I}_q$ are regular because of the non-zero elements in its main diagonal. The first subsystem is similar to the system $\boldsymbol{H}_q \boldsymbol{z}'_q(t) = \boldsymbol{z}_q(t)$ and hence:

$$\tilde{\boldsymbol{z}}_{p_0}(t) = \boldsymbol{0}_{1,p_0}.$$

The other subsystems have solutions:

$$\tilde{\boldsymbol{z}}_p(t) = \tilde{\boldsymbol{z}}_p(0)e^{\tilde{\boldsymbol{J}}_p t}, \quad \text{and} \quad \tilde{\boldsymbol{z}}_q(t) = \tilde{\boldsymbol{z}}_q(0)e^{\tilde{\boldsymbol{J}}_q t}.$$

Where

$$\tilde{\boldsymbol{J}}_p = (\boldsymbol{J}_p)^{-1}, \quad \tilde{\boldsymbol{J}}_q = \boldsymbol{H}_q.$$

By using $\boldsymbol{P}$ as defined in (6) we get:

$$\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{z}}(t)P = \begin{bmatrix} \boldsymbol{0}_{p_0,1} & e^{\tilde{\boldsymbol{J}}_p t}\tilde{\boldsymbol{z}}_p(0) & e^{\tilde{\boldsymbol{J}}_q t}\tilde{\boldsymbol{z}}_q(0) \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_{p_0} \\ \boldsymbol{P}_p \\ \boldsymbol{P}_q \end{bmatrix},$$

or, equivalently,

$$\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{z}}_p(0)e^{\tilde{\boldsymbol{J}}_p t}\boldsymbol{P}_p + \tilde{\boldsymbol{z}}_q(0)e^{\tilde{\boldsymbol{J}}_q t}\boldsymbol{P}_q,$$

or, equivalently,

$$\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{z}}_{p+q}(0)e^{\boldsymbol{J}_{p+q} t} \begin{bmatrix} \boldsymbol{P}_p \\ \boldsymbol{P}_q \end{bmatrix},$$

where $e^{\boldsymbol{J}_{p+q} t} = e^{\tilde{\boldsymbol{J}}_p t} \oplus e^{\tilde{\boldsymbol{J}}_q t}$, $\tilde{\boldsymbol{z}}_{p+q}(0) = \begin{bmatrix} \tilde{\boldsymbol{z}}_p(0) & \tilde{\boldsymbol{z}}_q(0) \end{bmatrix}$. This means that $\begin{bmatrix} \boldsymbol{P}_p \\ \boldsymbol{P}_q \end{bmatrix}$ has rows the left linear independent eigenvectors of the finite eigenvalues of $\tilde{s}\boldsymbol{A} - \boldsymbol{E}$. Hence:

$$\tilde{\boldsymbol{P}}_{\tilde{p}} = \begin{bmatrix} \boldsymbol{P}_p \\ \boldsymbol{P}_q \end{bmatrix}.$$

The proof is complete.

Remark 2.1. In (7) the matrices $\tilde{\boldsymbol{P}}_{\tilde{p}}$, $\tilde{\boldsymbol{Q}}_{\tilde{p}}$ are defined from eigenvectors related to eigenvalues of the pencil $s\boldsymbol{E} - \boldsymbol{A}$. Hence, it is worth mentioning that these matrices are not uniquely defined since only the span of the eigenvectors, i.e. eigenspace, is unique; any basis of this is a basis of eigenvectors to the given eigenvalue and may form the rows of the matrix.

3 | PRIME AND DUAL SYSTEMS

In this section we provide our main results. We will refer to system of FDEs (1) as the *prime* system and we define the system (2) as the *dual* system of (1). We will provide their solutions by using the three different definitions (3), (4), (5) by only using the spectrum of the pencil $s\boldsymbol{E} - \boldsymbol{A}$ of a first order system of differential equations.We prove the following Theorem:

Theorem 3.1. Let $\boldsymbol{Q}_{p_0}$, $\boldsymbol{Q}_p$, and $\boldsymbol{Q}_q$ be matrices with columns the right eigenvectors of the zero eigenvalue, the non-zero finite eigenvalues, and the infinite eigenvalue respectively of the pencil $s\boldsymbol{E} - \boldsymbol{A}$ as defined in (6). Then:

(a) The general solution of the prime system of FDEs (1) is given by

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p \end{bmatrix} \left[\boldsymbol{\Phi}_0(t) \oplus \boldsymbol{\Phi}(t)\right] \boldsymbol{c}. \tag{9}$$

Where $\boldsymbol{\Phi}_0(t)$, $\boldsymbol{\Phi}(t)$ are given as follows:

(i) If we use the (C) fractional derivative:

$$\boldsymbol{\Phi}_0(t) = \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ka+1)} \boldsymbol{J}_{p_0}^k, \quad \boldsymbol{\Phi}(t) = \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ka+1)} \boldsymbol{J}_p^k. \tag{10}$$

(ii) If we use the (CF) fractional derivative:

$$\begin{aligned} \boldsymbol{\Phi}_0(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} (1-a)^n a^{k-n} \frac{t^{k-n}}{\Gamma(k+1-n)} \boldsymbol{J}_{p_0}^k, \\ \boldsymbol{\Phi}(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} (1-a)^n a^{k-n} \frac{t^{k-n}}{\Gamma(k+1-n)} \boldsymbol{J}_{p}^k. \end{aligned} \tag{11}$$

(iii) If we use the (AB) fractional derivative:

$$\begin{aligned} \boldsymbol{\Phi}_0(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{(1-a)^n a^{k-n}}{B^k(a)} \frac{t^{ak+2-an}}{\Gamma(ak+1-an)} \boldsymbol{J}_{p_0}^k, \\ \boldsymbol{\Phi}(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{(1-a)^n a^{k-n}}{B^k(a)} \frac{t^{ak+2-an}}{\Gamma(ak+1-an)} \boldsymbol{J}_{p}^k. \end{aligned} \tag{12}$$

In addition, for given initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$ the solution is unique if and only if:

$$\boldsymbol{x}_0 \in colspan \begin{bmatrix} \boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p \end{bmatrix}. \tag{13}$$

The unique solution is then given by (9) and $\boldsymbol{c}$ is the unique solution of the linear system

$$\begin{bmatrix} \boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p \end{bmatrix} \boldsymbol{c} = \boldsymbol{x}_0. \tag{14}$$

(b) The general solution of the dual system (2) is given by

$$\tilde{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix} \left[\boldsymbol{\Psi}_0(t) \oplus \boldsymbol{\Psi}(t) \right] \boldsymbol{c}. \tag{15}$$

Where $\boldsymbol{\Psi}_0(t)$, $\boldsymbol{\Psi}(t)$ are given as follows:

(i) If we use the (C) fractional derivative:

$$\boldsymbol{\Psi}_0(t) = \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ka+1)} \boldsymbol{H}_q^k, \quad \boldsymbol{\Psi}(t) = \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ka+1)} [\boldsymbol{J}_p^{-1}]^k. \tag{16}$$

(ii) If we use the (CF) fractional derivative:

$$\begin{aligned} \boldsymbol{\Psi}_0(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} (1-a)^n a^{k-n} \frac{t^{k-n}}{\Gamma(k+1-n)} \boldsymbol{H}_{q}^k, \\ \boldsymbol{\Psi}(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} (1-a)^n a^{k-n} \frac{t^{k-n}}{\Gamma(k+1-n)} [\boldsymbol{J}_{p}^{-1}]^k. \end{aligned} \tag{17}$$

(iii) If we use the (AB) fractional derivative:

$$\begin{aligned} \boldsymbol{\Psi}_0(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{(1-a)^n a^{k-n}}{B^k(a)} \frac{t^{ak+2-an}}{\Gamma(ak+1-an)} \boldsymbol{H}_{q}^k, \\ \boldsymbol{\Psi}(t) &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{(1-a)^n a^{k-n}}{B^k(a)} \frac{t^{ak+2-an}}{\Gamma(ak+1-an)} [\boldsymbol{J}_{p}^{-1}]^k. \end{aligned} \tag{18}$$

In addition, for given initial conditions $\tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0$ the solution is unique if and only if:

$$\tilde{\boldsymbol{x}}_0 \in colspan \begin{bmatrix} \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix}. \tag{19}$$

The unique solution is then given by (15) and $\boldsymbol{c}$ is the unique solution of the linear system

$$\begin{bmatrix} \boldsymbol{Q}_p & \boldsymbol{Q}_q \end{bmatrix} \boldsymbol{c} = \tilde{\boldsymbol{x}}_0. \tag{20}$$

Where $\boldsymbol{J}_{p_0}$, $\boldsymbol{J}_p$, and $\boldsymbol{H}_q$ are Jordan matrices of the zero eigenvalue, the non-zero finite eigenvalues, and the infinite eigenvalue respectively of the pencil $s\boldsymbol{E} - \boldsymbol{A}$.

Proof. As discussed in section 2, there exist invertible matrices $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{C}^{m\times m}$ for $s\boldsymbol{E}-\boldsymbol{A}$ as defined in (6). If we substitute $\boldsymbol{x}(t)=\boldsymbol{Q}\boldsymbol{z}(t)$ into (1) and multiply by $\boldsymbol{P}$ using (6)we have:

$$\boldsymbol{E}\boldsymbol{Q}\,{}_0D^a_t\boldsymbol{z}(t)=\boldsymbol{A}\boldsymbol{Q}\boldsymbol{z}(t),$$

and

$$\left[\boldsymbol{I}_{p_0}\oplus\boldsymbol{I}_p\oplus\boldsymbol{H}_q\right]\begin{bmatrix}\boldsymbol{z}^{(a)}_{p_0}(t)\\ \boldsymbol{z}^{(a)}_{p}(t)\\ \boldsymbol{z}^{(a)}_{q}(t)\end{bmatrix}=\left[\boldsymbol{J}_{p_0}\oplus\boldsymbol{J}_p\oplus\boldsymbol{I}_q\right]\begin{bmatrix}\boldsymbol{z}_{p_0}(t)\\ \boldsymbol{z}_{p}(t)\\ \boldsymbol{z}_{q}(t)\end{bmatrix}.$$

Where

$$\boldsymbol{z}(t)=\begin{bmatrix}\boldsymbol{z}_{p_0}(t)\\ \boldsymbol{z}_{p}(t)\\ \boldsymbol{z}_{q}(t)\end{bmatrix},$$

with $\boldsymbol{z}_{p_0}(t)\in\mathbb{C}^{p_0\times1}$, $\boldsymbol{z}_p(t)\in\mathbb{C}^{p\times1}$, $\boldsymbol{z}_q(t)\in\mathbb{C}^{q\times1}$. Consequently:

$$\boldsymbol{z}^{(a)}_{p_0}(t)=\boldsymbol{J}_{p_0}\boldsymbol{z}_p(t),$$

$$\boldsymbol{z}^{(a)}_{p}(t)=\boldsymbol{J}_{p}\boldsymbol{z}_p(t),$$

$$\boldsymbol{H}_q\boldsymbol{z}^{(a)}_{q}(t)=\boldsymbol{z}_q(t).$$

For the third subsystem we have:

$$\boldsymbol{z}_q(t)=\boldsymbol{0}_{q,1}.$$

This can be proved as follows. If we denote ${}_0D^{ka}_t$ with ${}^{(}ka)$, $k\in\mathbb{N}^*$ and q_* is index of $\boldsymbol{H}_q$ such that $H^{q_*}_q=\boldsymbol{0}_{q,q}$, then if we obtain the following matrix equations

$$\begin{array}{c}\boldsymbol{H}_q\boldsymbol{z}^{(a)}_q(t)=\boldsymbol{z}_q(t)\\ \boldsymbol{H}^2_q\boldsymbol{z}^{(2a)}_q(t)=\boldsymbol{H}_q\boldsymbol{z}^{(a)}_q(t)\\ \boldsymbol{H}^3_q\boldsymbol{z}^{(3a)}_q(t)=\boldsymbol{H}^2_q\boldsymbol{z}^{(2a)}_q(t)\\ \boldsymbol{H}^4_q\boldsymbol{z}^{(4a)}_q(t)=\boldsymbol{H}^3_q\boldsymbol{z}^{(3a)}_q(t)\\ \vdots\\ \boldsymbol{H}^{q_*-1}_q\boldsymbol{z}^{([q_*-1]a)}_q(t)=\boldsymbol{H}^{q_*-2}_q\boldsymbol{z}^{([q_*-2]a)}_q(t)\\ \boldsymbol{H}^{q_*}_q\boldsymbol{z}^{(q_*a)}_q(t)=\boldsymbol{H}^{q_*-1}_q\boldsymbol{z}^{([q_*-1]a)}_q(t)\end{array},$$

by taking their sum we arrive easily at the solution. For the second subsystem if we use the Laplace transform $\mathcal{L}\{\boldsymbol{z}_p(t)\}=\boldsymbol{w}(s)$:

$$\mathcal{L}\{{}_0D^a_t\boldsymbol{z}_p(t)\}=\boldsymbol{J}_p\mathcal{L}\{\boldsymbol{z}_p(t)\},$$

or, equivalently,

$$z\boldsymbol{w}(s)-w\boldsymbol{z}_p(0)=\boldsymbol{J}_p\boldsymbol{w}(s).$$

Where

(i) $z=s^a$, $w=s^{a-1}$, for ${}_0D^a_t:={}^C_0D^a_t$;

(ii) $z=\frac{s}{s+a(1-s)}$, $w=\frac{1}{s+a(1-s)}$, for ${}_0D^a_t:={}^{CF}_0D^a_t$;

(iii) $z=\frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}$, $w=\frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}$, for ${}_0D^a_t:={}^{ABC}_0D^a_t$.

By setting $\boldsymbol{z}_{p0}=\boldsymbol{c}$:

$$(z\boldsymbol{I}_p-\boldsymbol{J}_p)\boldsymbol{w}(s)=w\boldsymbol{c},$$

or, equivalently,

$$\boldsymbol{w}(s)=w(z\boldsymbol{I}_p-\boldsymbol{J}_p)^{-1}\boldsymbol{c},$$

or, equivalently, by taking into account that $(z\boldsymbol{I}_p-\boldsymbol{J}_p)^{-1}=\sum_{k=0}^{\infty}z^{-k-1}\boldsymbol{J}^k_p$,

$$\boldsymbol{w}(s)=\sum_{k=0}^{\infty}wz^{-k-1}\boldsymbol{J}^k_p\boldsymbol{c}.$$

We have the following cases:

(a) For ${}_0D_t^a := {}_0^CD_t^a$, by replacing in the above expression $z = s^a$, $w = s^{a-1}$ we have:

$$\boldsymbol{w}(s) = \sum_{k=0}^{\infty} s^{-ak-1}\boldsymbol{J}_p^k\boldsymbol{c}_1.$$

By using (10) we have

$$\boldsymbol{z}_p(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}_1.$$

Similarly for the first subsystem if we use (10) the solution for the (C) fractional derivative is

$$\boldsymbol{z}_{p_0}(t) = \boldsymbol{\Phi}_0(t)\boldsymbol{c}_0.$$

(b) For ${}_0D_t^a := {}_0^{CF}D_t^a$, by replacing $z = \frac{s}{s+a(1-s)}$, $w = \frac{1}{s+a(1-s)}$ we have

$$\boldsymbol{w}(s) = \sum_{k=0}^{\infty} \frac{1}{s+a(1-s)}\left[\frac{s}{s+a(1-s)}\right]^{-k-1}\boldsymbol{J}_p^k\boldsymbol{c}_1.$$

Equally:

$$\boldsymbol{w}(s) = \sum_{k=0}^{\infty} \frac{[(1-a)s+a]^k}{s^{k+1}}\boldsymbol{J}_p^k\boldsymbol{c}_1,$$

and consequently, since $[(1-a)s+a]^k = \sum_{n=0}^{k}\binom{k}{n}(1-a)^n s^n a^{k-n}$

$$\boldsymbol{w}(s) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}(1-a)^n s^{n-k-1}a^{k-n}\boldsymbol{J}_p^k\boldsymbol{c}_1.$$

Using (11) we have

$$\boldsymbol{z}_p(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}_1.$$

Similarly for the first subsystem if we use (11) the solution for the (CF) fractional derivative is

$$\boldsymbol{z}_{p_0}(t) = \boldsymbol{\Phi}_0(t)\boldsymbol{c}_0.$$

(c) For ${}_0D_t^a := {}^{ABC}_{\ \ 0}D_t^a$, by replacing $z = \frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}$, $w = \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}$ we have

$$\boldsymbol{w}(s) = \sum_{k=0}^{\infty} \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}\left[\frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}\right]^{-k-1}\boldsymbol{J}_p^k\boldsymbol{c}_1.$$

Equally:

$$\boldsymbol{w}(s) = \sum_{k=0}^{\infty} \frac{[(1-a)s^a+a]^k}{B^k(a)s^{ak+1}}\boldsymbol{J}_p^k\boldsymbol{c}_1,$$

and consequently, since $[(1-a)s^a+a]^k = \sum_{n=0}^{k}\binom{k}{n}(1-a)^n s^{an}a^{k-n}$

$$\boldsymbol{w}(s) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}\frac{(1-a)^n a^{k-n}}{B^k(a)}s^{an-ak-1}\boldsymbol{J}_p^k\boldsymbol{c}_1.$$

Using (12) we have

$$\boldsymbol{z}_p(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}_1.$$

Similarly for the first subsystem if we use (12) the solution for the (AB) fractional derivative is

$$\boldsymbol{z}_{p_0}(t) = \boldsymbol{\Phi}_0(t)\boldsymbol{c}_0.$$

To conclude, by using the above solutions of the three subsystems we obtain the following general solution for the system of FDEs (2):

$$\boldsymbol{x}(t) = \boldsymbol{Q}\boldsymbol{z}(t) = \begin{bmatrix}\boldsymbol{Q}_{p_0} & \boldsymbol{Q}_p & \boldsymbol{Q}_q\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}_0(t)\boldsymbol{c}_0\\ \boldsymbol{\Phi}(t)\boldsymbol{c}_1\\ \boldsymbol{0}_{q,1}\end{bmatrix},$$

or, equivalently,

$$\boldsymbol{x}(t) = \boldsymbol{Q}_{p_0}\boldsymbol{\Phi}_0(t)\boldsymbol{c}_0 + \boldsymbol{Q}_p\boldsymbol{\Phi}(t)\boldsymbol{c}_1,$$

which leads to (9). For given initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$ if we use (9) for $t = 0$ we get (14). Hence $\boldsymbol{c}$ in (9) can be uniquely defined if and only if the algebraic system (14) has a unique solution which happens if and only if condition (13) holds. Then because $\left[\, \boldsymbol{Q}_{p_0} \;\; \boldsymbol{Q}_p \,\right]$ has linear independent columns, the linear algebraic system (14) has always a unique solution in respect to $\boldsymbol{c}$.

Next, we consider the dual system (2). Similarly to the prime system by applying $\tilde{\boldsymbol{x}} = \boldsymbol{Q}\tilde{\boldsymbol{z}}$ into (2) we arrive at three subsystems of the following form:

$$\boldsymbol{J}_{p_0}\,{}_0D_t^a\tilde{\boldsymbol{z}}_{p_0}(t) = \tilde{\boldsymbol{z}}_p(t),$$

$$\boldsymbol{J}_p\,{}_0D_t^a\tilde{\boldsymbol{z}}_p(t) = \tilde{\boldsymbol{z}}_p(t),$$

$${}_0D_t^a\tilde{\boldsymbol{z}}_q(t) = \boldsymbol{H}_q\tilde{\boldsymbol{z}}_q(t).$$

or, equivalently,

$$\boldsymbol{H}_{p_0}\,{}_0D_t^a\tilde{\boldsymbol{z}}_{p_0}(t) = \tilde{\boldsymbol{z}}_p(t),$$

$${}_0D_t^a\tilde{\boldsymbol{z}}_p(t) = J_p^{-1}\tilde{\boldsymbol{z}}_p(t),$$

$${}_0D_t^a\tilde{\boldsymbol{z}}_q(t) = \boldsymbol{H}_q\tilde{\boldsymbol{z}}_q(t).$$

Note that $\boldsymbol{J}_{p_0} = \boldsymbol{H}_{p_0}$, and $\boldsymbol{J}_p$ is regular because of non-zero elements in its main diagonal. Similarly to the solutions of the subsystems obtained in the case of the prime system:

$$\tilde{\boldsymbol{z}}_{p_0}(t) = \boldsymbol{0}_{p_0},$$

while the other two subsystems have the following general solutions:

$$\tilde{\boldsymbol{z}}_p(t) = \boldsymbol{\Psi}_0(t)\boldsymbol{c}_0, \quad \tilde{\boldsymbol{z}}_q(t) = \boldsymbol{\Psi}(t)\boldsymbol{c}_1,$$

with $\boldsymbol{\Psi}_0(t)$, $\boldsymbol{\Psi}(t)$ being defined in (16), (17), (18). Furthermore from Theorem 2.1, $\left[\, \boldsymbol{Q}_p \;\; \boldsymbol{Q}_q \,\right]$ has the eigenvectors of all finite eigenvalues of $\tilde{s}\boldsymbol{A} - \boldsymbol{E}$. Hence similarly to the prime system, the general solution of the dual system (2) is given by (15). For given initial conditions $\tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0$ if we use (15) for $t = 0$ we get (20). Hence $\boldsymbol{c}$ in (15) can be uniquely defined if and only if the algebraic system (20) has a unique solution which happens if and only if condition (19) holds. The matrix $\left[\, \boldsymbol{Q}_p \;\; \boldsymbol{Q}_q \,\right]$ has linear independent columns and hence the linear algebraic system (20) has a unique solution in respect to $\boldsymbol{c}$. The proof is completed.

4 | EXAMPLES AND COMPUTATIONAL RESULTS

In this Section we consider the prime system of FDEs (1), its dual system (2) for $a = 0.5$ and present two examples. We apply Theorem 3.1 and also include a computational analysis in the software Modelica.

Example 1

Let

$$\boldsymbol{E} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Let $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\tilde{\boldsymbol{x}}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ be initial conditions. Then

$$s\boldsymbol{E} - \boldsymbol{A} = \begin{bmatrix} s-1 & s-1 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of $s\boldsymbol{E} - \boldsymbol{A}$ are $\lambda_1 = 1$ with eigenspace

$$\left\langle \boldsymbol{u}_1 \right\rangle = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle,$$

and λ_∞, an infinite eigenvalue of algebraic multiplicity $q = 1$ with eigenspace:

$$\left\langle \boldsymbol{u}_2 \right\rangle = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle.$$

The Jordan matrices related to the eigenvalues are:

$$J_p = 1\ ,\ H_q = 0.$$

Furthermore $\boldsymbol{Q}_p = [\boldsymbol{u}_1], \boldsymbol{Q}_q = [\boldsymbol{u}_2]$. Since there is no zero eigenvalue, $\boldsymbol{Q}_{p_0}$ does not exist. If we use the (C) fractional derivative the solution is given by (9), (10):

$$\boldsymbol{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sum_{k=0}^{\infty} \frac{t^{0.5k}}{\Gamma(k0.5+1)} \left[1^k\right] c,$$

or, equivalently,

$$\boldsymbol{x}(t) = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^{0.5k} c}{\Gamma(0.5k+1)} \\ 0 \end{bmatrix}.$$

In addition, it is easy to observe that $\boldsymbol{x}_0 \in colspan\boldsymbol{Q}_p$, i.e. (13) does not hold, and hence by using (14) we have $c = 1$. Thus:

$$\boldsymbol{x}(t) = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^{0.5k}}{\Gamma(0.5k+1)} \\ 0 \end{bmatrix}.$$

Similarly if we use the (CF) fractional derivative given by (11), (12) we have:

$$\boldsymbol{x}(t) = \begin{bmatrix} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} (0.5)^k \frac{t^{k-n}}{\Gamma(k+1-n)} \\ 0 \end{bmatrix}.$$

Finally, the general solution of the prime system (1) if we use the (AB) fractional derivative is given by (13), (14):

$$\boldsymbol{x}(t) = \begin{bmatrix} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{(0.5)^k}{B^k(0.5)} \frac{t^{0.5k+2-0.5n}}{\Gamma(0.5k+1-0.5n)} \\ 0 \end{bmatrix}.$$

The general solution of the dual system (2) is given by (15), (16): and if we set $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$ we have:

$$\tilde{\boldsymbol{x}}(t) = \sum_{k=0}^{\infty} \frac{t^{0.5k}}{\Gamma(0.5k+1)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

It is easy to observe that (19) holds, and hence the system has a unique solution. By using (20) we have

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Hence $c_1 = c_2 = 1$, and the unique solution of (2) is given by:

$$\tilde{\boldsymbol{x}}(t) = \sum_{k=0}^{\infty} \frac{t^{0.5k}}{\Gamma(0.5k+1)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Similarly if we use the (CF) fractional derivative, the solution of the dual system is given by (15), (17):

$$\tilde{\boldsymbol{x}}(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} 0.5^k \frac{t^{k-n}}{\Gamma(k+1-n)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Finally, the general solution of the dual system (2) if we use the (AB) fractional derivative is given by (15), (18):

$$\tilde{\boldsymbol{x}}(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{0.5^k}{B^k(0.5)} \frac{t^{0.5k+2-0.5n}}{\Gamma(0.5k+1-0.5n)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Example 2

For our second example, let $a = 0.5$ in (1), (2), and:

$$\boldsymbol{E} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ -4&2&2&-3&-2&-1\\ 1&1&-1&-1&0&0 \end{bmatrix}.$$

Also let $\boldsymbol{x}_0 = \begin{bmatrix} -4 & 6 & -5 & 7 & -7 & 9 \end{bmatrix}^T$, $\tilde{\boldsymbol{x}}_0 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^T$, be initial conditions. The pencil $s\boldsymbol{E} - \boldsymbol{A}$ has the eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, of algebraic multiplicity $p_1 = p_2 = p_3 = 1$, and eigenspaces

$$\left\langle \boldsymbol{u}_1 \right\rangle = \left\langle \begin{bmatrix} -1\\1\\-3\\3\\-9\\9 \end{bmatrix} \right\rangle, \quad \left\langle \boldsymbol{u}_2 \right\rangle = \left\langle \begin{bmatrix} -1\\1\\-2\\2\\-4\\4 \end{bmatrix} \right\rangle, \quad \left\langle \boldsymbol{u}_3 \right\rangle = \left\langle \begin{bmatrix} -3\\5\\-3\\5\\-3\\5 \end{bmatrix} \right\rangle,$$

and λ_∞ (infinite eigenvalue) of algebraic multiplicity $q = 3$ and eigenspace:

$$\left\langle \boldsymbol{u}_4, \quad \boldsymbol{u}_5, \quad \boldsymbol{u}_6 \right\rangle = \left\langle \begin{bmatrix} 0\\0\\0\\0\\-1\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\-1\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} -1\\1\\0\\1\\42\\-48 \end{bmatrix} \right\rangle.$$

The Jordan matrices are:

$$\boldsymbol{J}_p = \begin{bmatrix} 3&0&0\\ 0&2&0\\ 0&0&1 \end{bmatrix}, \quad \boldsymbol{H}_q = \begin{bmatrix} 0&1&0\\ 0&0&1\\ 0&0&0 \end{bmatrix}.$$

Since there is no zero eigenvalue, $\boldsymbol{Q}_{p_0}$ does not exist. The matrices $\boldsymbol{Q}_p$, $\boldsymbol{Q}_q$ are defined as $\boldsymbol{Q}_p = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{bmatrix}$, $\boldsymbol{Q}_q = \begin{bmatrix} \boldsymbol{u}_4 & \boldsymbol{u}_5 & \boldsymbol{u}_6 \end{bmatrix}$ respectively. The general solution of the prime system (1) is given by (9), (10):

$$\boldsymbol{x}(t) = \begin{bmatrix} -1&-1&-3\\ 1&1&5\\ -3&-2&-3\\ 3&2&5\\ -9&-4&-3\\ 9&4&5 \end{bmatrix} \sum_{k=0}^{\infty} \frac{t^{0.5k}}{\Gamma(0.5k+1)} \begin{bmatrix} 3^k&0&0\\ 0&2^k&0\\ 0&0&1 \end{bmatrix} \boldsymbol{c}.$$

or, equivalently, if we set $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$,

$$\boldsymbol{x}(t) = \sum_{k=0}^{\infty} \frac{t^{0.5k}}{\Gamma(0.5k+1)} \begin{bmatrix} -3^k c_1 - 2^k c_2 - 3c_3\\ 3^k c_1 + 2^k c_2 + 5c_3\\ -3^{k+1} c_1 - 2^{k+1} c_2 - 3c_3\\ 3^{k+1} c_1 + 2^{k+1} c_2 + 5c_3\\ -3^{k+2} c_1 - 2^{k+2} c_2 - 3c_3\\ 3^{k+2} c_1 + 2^{k+2} c_2 + 5c_3 \end{bmatrix}.$$

In addition, it is easy to observe that $\boldsymbol{x}_0 \in colspan\boldsymbol{Q}_p$, and hence using (14) we have $c_1 = 0, c_2 = c_3 = 1$, and the unique solution of the prime system is given by:

$$\boldsymbol{x}(t) = \sum_{k=0}^{\infty} \frac{t^{0.5k}}{\Gamma(0.5k+1)} \begin{bmatrix} -2^k - 3 \\ 2^k + 5 \\ -2^{k+1} - 3 \\ 2^{k+1}c + 5 \\ -2^{k+2} - 3 \\ 2^{k+2} + 5 \end{bmatrix}.$$

Similarly if we use the (CF) fractional derivative given by (11), (12) we have:

$$\boldsymbol{x}(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} 0.5^k \frac{t^{k-n}}{\Gamma(k+1-n)} \begin{bmatrix} -2^k - 3 \\ 2^k + 5 \\ -2^{k+1} - 3 \\ 2^{k+1}c + 5 \\ -2^{k+2} - 3 \\ 2^{k+2} + 5 \end{bmatrix}.$$

Finally, the general solution of the prime system (1) if we use the (AB) fractional derivative is given by (13), (14):

$$\boldsymbol{x}(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{0.5^k}{B^k(0.5)} \frac{t^{0.5k+2-0.5n}}{\Gamma(0.5k+1-0.5n)} \begin{bmatrix} -2^k - 3 \\ 2^k + 5 \\ -2^{k+1} - 3 \\ 2^{k+1}c + 5 \\ -2^{k+2} - 3 \\ 2^{k+2} + 5 \end{bmatrix}.$$

The general solution of the dual system (2) is given by (15), (16):

$$\tilde{\boldsymbol{x}} = \sum_{k=2}^{\infty} \frac{t^{0.5k}}{\Gamma(0.5k+1)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2^k} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2^k} & 1 \\ 0 & 0 & 0 & -\frac{1}{3^k} & 0 & 42 \\ 0 & 0 & 0 & \frac{1}{3^k} & \frac{1}{2^k} & -48 \end{bmatrix} \boldsymbol{c}$$

It is easy to observe that $\tilde{\boldsymbol{x}}_0 \notin colspan\left[\boldsymbol{Q}_p \ \boldsymbol{Q}_q\right]$, i.e. (19) does not hold, and hence the system does not have a unique solution.

Similarly if we use the (CF) fractional derivative, the solution of the dual system is given by (15), (17):

$$\tilde{\boldsymbol{x}} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} 0.5^k \frac{t^{k-n}}{\Gamma(k+1-n)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2^k} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2^k} & 1 \\ 0 & 0 & 0 & -\frac{1}{3^k} & 0 & 42 \\ 0 & 0 & 0 & \frac{1}{3^k} & \frac{1}{2^k} & -48 \end{bmatrix} \boldsymbol{c}.$$

Finally, the general solution of the dual system (2) if we use the (AB) fractional derivative is given by (15), (18):

$$\tilde{\boldsymbol{x}} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \frac{0.5^k}{B^k(0.5)} \frac{t^{0.5k+2-0.5n}}{\Gamma(0.5k+1-0.5n)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2^k} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2^k} & 1 \\ 0 & 0 & 0 & -\frac{1}{3^k} & 0 & 42 \\ 0 & 0 & 0 & \frac{1}{3^k} & \frac{1}{2^k} & -48 \end{bmatrix} \boldsymbol{c}.$$

Examples in Modelica

In this subsection, based on the proposed matrix polynomial of first order and an integer-order approximation method, we will provide examples on the solutions of the system of FDEs (1) implemented in the open-source Modelica (OPENMODELICA), see[26]. We will use the fractional derivative (C). The integer–order approximation for the fractional derivative by Oustaloup's method in the Laplace domain, see[27], is given by:

$$s^a \approx \omega_h^a \prod_{k=-N}^{N} \frac{s+\omega_k'}{s+\omega_k}. \tag{21}$$

Where (ω_b, ω_h) is the fitting range, a is the order of the fractional derivative, N is order of the approximation, and

$$\omega_k = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+0.5(1+a)}{2N+1}}, \quad \omega_k' = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+0.5(1-a)}{2N+1}}.$$

Figure 1 shows the Bode plot with approximation order $N = 4$ and frequency range $\omega_b = 0.001$ Hz, $\omega_h = 1000$ Hz for a first order system of differential equations (left) and a system of FDEs of order $a = 0.5$ (right). The integrators have a frequency response with a slope of -20dB and -10 dB/Decade; -90 and -45 degree phase angle respectively. Observe that within the chosen fitting range the slope of the amplitude shows an acceptable fit. In the following examples we use this approximation order and fitting range.

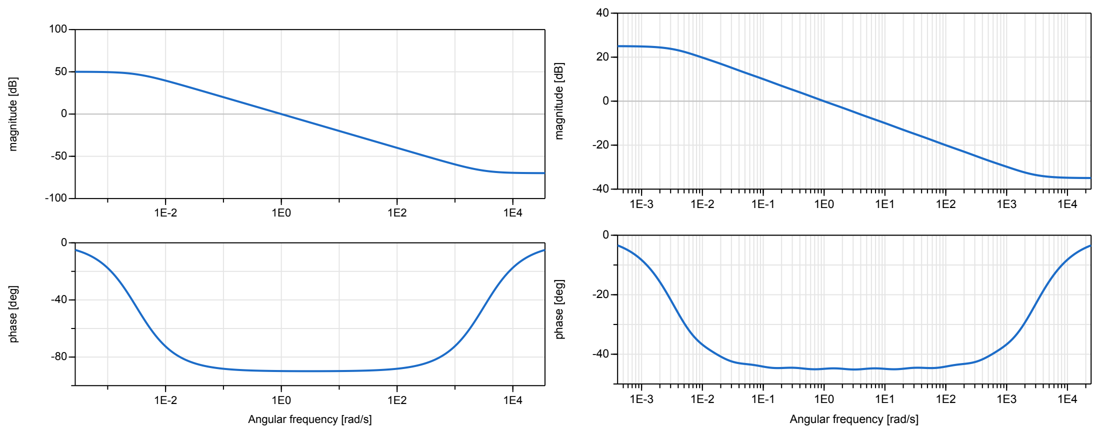

FIGURE 1 On the left Bode plot of integrator for a first order system of differential equations & on the right Bode plot of integrator for a system of FDEs of order $a = 0.5$. We use a 4th order approximation and fitting range (0.001 Hz, 1000 Hz).

The following examples are simulated using two Modelica based simulation tools: Dymola, see[15], and OpenModelica, see[26]. We firstly consider the system $\boldsymbol{E}\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$ with initial conditions $\boldsymbol{x}_0 = [-1 \ -1]^T$ and

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{22}$$

Since this system is linear and of first order, plotting the numerical solution in Modelica, with and without Ostaloup approximation should be identical. Note that $x_1 = x_2$. The simulation results are shown in Fig. 2 on the left. It can be observed that indeed the results are identical. Next we assume the system of FDEs (1) with $\boldsymbol{E}$, $\boldsymbol{A}$ given in (22) and simulate this example considering three different fractional orders $a = 0.5, 0.6$ and 0.8. Note as previously $x_1 = x_2$. The simulation results are shown in Fig. 2 on the right.

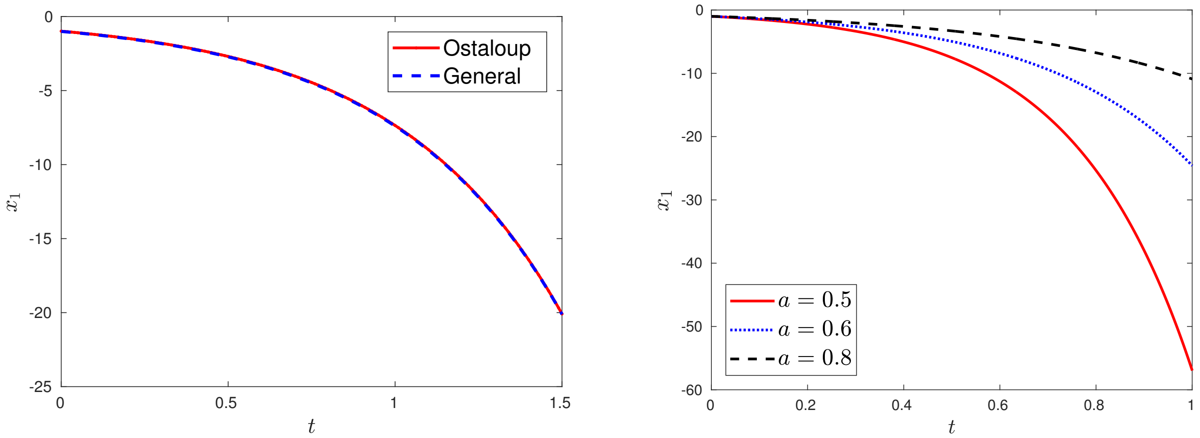

***FIGURE 2** On the left trajectories of solutions for the first order system by using its general solution and the Ostaloup approximation. On the right trajectories of the system of FDEs using the Ostaloup approximation for $a = 0.5, 0.6$ and 0.8.*

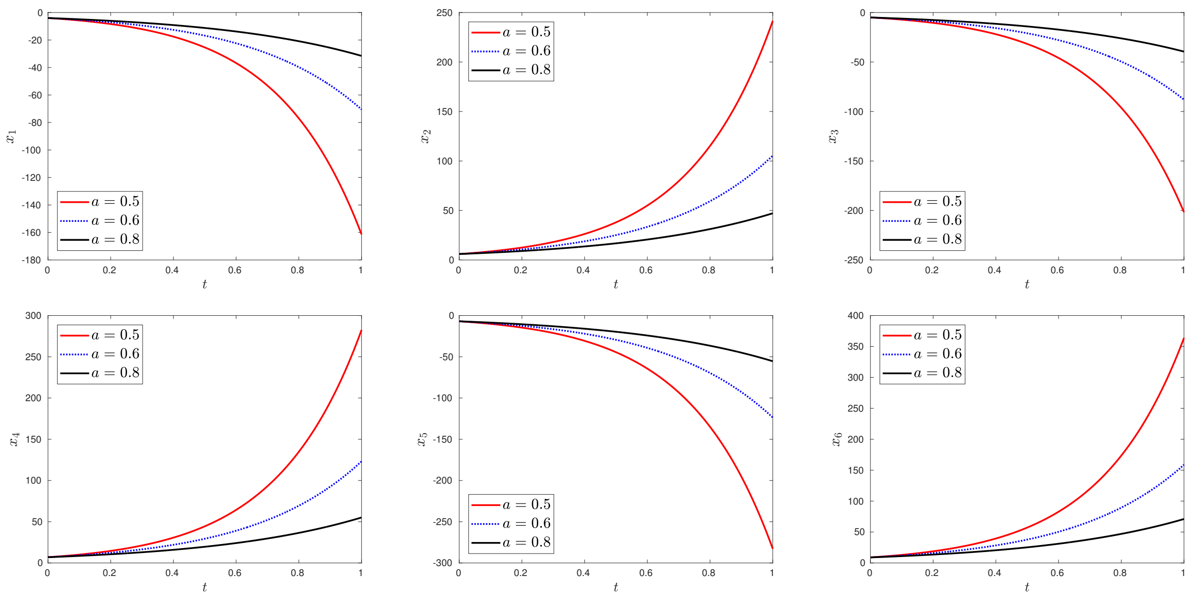

***FIGURE 3** Trajectories of the second example using the Ostaloup approximation for three different fractional orders $a = 0.5, 0.6$ and 0.8.*

We consider now system (1) with initial conditions $\boldsymbol{x}_0 = [-4 \;\; 6 \;\; -5 \;\; 7 \;\; -7 \;\; 9]^T$ and

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 2 & 2 & -3 & -2 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix}. \tag{23}$$

For three different cases $a = 0.5, 0.6$ and 0.8 the simulation results are shown in Figs. 3 while a comparison of the trajectories of the second example are shown in Figs. 4

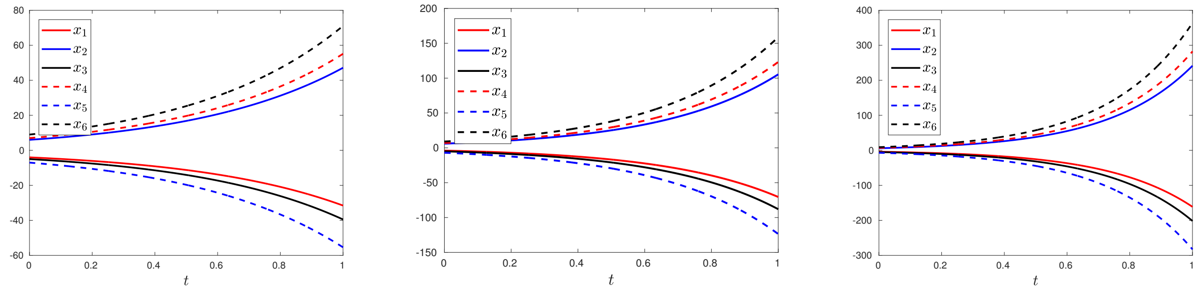

FIGURE 4 *A comparison of the trajectories of the second example using the Ostaloup approximation for three different fractional orders a* = 0.5, 0.6 *and* 0.8.

CONCLUSIONS

In this article we used three different fractional operators, the (C), (CF), (ABC), and considered a class of systems of FDEs and its dual systems. We proved that by only using the spectrum of a linear pencil, a matrix polynomial of first order, and not the fractional pencil of the prime fractional system, we can study the solutions for the prime, and its dual system without additional computational cost. Numerical examples and computational results on the software modelica concluded the article.

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