On the Stability Analysis of Systems of Neutral Delay Differential Equations

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Abstract. This paper focuses on the stability analysis of systems modeled as Neutral Delay Differential Equations (NDDEs). These systems include delays in both the state variables and their time derivatives. The proposed approach consists of a descriptor model transformation that constructs an equivalent set of Delay Differential Algebraic Equations (DDAEs) of the original NDDEs. We first rigourously prove the equivalency between the original set of NDDEs and the transformed set of DDAEs. Then, the effect on stability analysis is evaluated numerically through a delay-independent stability criterion and the Chebyshev discretization of the characteristic equations.

Keywords: Time delay, delay differential algebraic equations (DDAEs), neutral time-delay differential equations (NDDEs), Eigenvalue analysis, delay-independent stable.

1 Introduction

Neutral Time-Delay Differential Equations (NDDEs) are systems where the delays appear in both the state variables and their time derivatives. They have wide applications in applied mathematics [1], physics [27], ecology [16] and engineering [22]. In this paper, we are interested

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in the evaluation of the stability of systems of NDDEs in the following form:

$$\mathbf{0}_{p,1} = \mathbf{f}(\mathbf{x}, \mathbf{x}(t-\tau), \dot{\mathbf{x}}, \dot{\mathbf{x}}(t-\tau)), \quad (1)$$

where $\mathbf{f}$ ($\mathbf{f} : \mathbb{R}^{4p} \mapsto \mathbb{R}^{p}$) are the differential equations and $\mathbf{x} = \mathbf{x}(t)$ ($\mathbf{x} \in \mathbb{R}^{p}$) are the state variables. Note that we include the case that $\mathbf{f}$ can be implicit with its partial derivatives to be singular matrices, i.e. $\det\big(\frac{\partial \mathbf{f}}{\partial \mathbf{k}}\big) = 0$, $\mathbf{k} = \mathbf{x}, \mathbf{x}(t-\tau), \dot{\mathbf{x}}, \dot{\mathbf{x}}(t-\tau)$ and $\dot{\mathbf{x}}, \dot{\mathbf{x}}(t-\tau)$ not to be zero columns. With $\mathbf{0}_{i,j}$ we denote the zero matrix of i rows and j columns.

Conventional approaches for the stability analysis of (1) are based on Lyapunov Functional Method (LFM) [7, 18, 19, 26]. These techniques require the solution of a Linear Matrix Inequality (LMI) problem. The complexity to construct the Lyapunov function and the heavy computational burden to solve the LMI problem limit the application of LFMs on engineering fields. Moreover, as LFMs provide only sufficient but not necessary conditions for system stability, they tend to be conservative.

There also exists a variety of frequency-domain approaches to solve the stability of Delay Differential Equations (DDEs)[2, 3, 8, 23, 29]. Most of these techniques are based on the solution of an eigenvalue problem. This consists in estimating the dominant modes of the DDEs through the solution of the characteristic equation of the system. In [12–14], we have developed a general eigenvalue analysis approach is developed to solve the stability of large system described by a set of Delay Differential Algebraic Equations (DDAEs). Compared to LFMs, eigenvalue-based approaches are less computationally intensive and provide a more accurate stability analysis. Note that the eigenvalue analysis requires a linear or linearized system. The results of the stability analysis are global if the original system is linear. For nonliner systems, only a local stability analysis can be drawn. This is still very relevant for many engineering systems and applications, e.g., the small-signal analysis of electrical energy systems [24].

This paper aims at developing a systematic stability solution of NDDEs in frequency-domain field.

Definition 1.1. Consider the system of DDAEs:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{g}(\mathbf{x}, \mathbf{x}(t-\tau), \mathbf{y}, \mathbf{y}(t-\tau)) \\ &, \\ \mathbf{0}_{q,1} &= \mathbf{h}(\mathbf{x}, \mathbf{x}(t-\tau), \mathbf{y}, \mathbf{y}(t-\tau))\end{aligned} \tag{2}$$

where $\mathbf{g}$ ($\mathbf{g} : \mathbb{R}^{2p+2q} \mapsto \mathbb{R}^p$) are the differential equations; $\mathbf{h}(\mathbf{h} : \mathbb{R}^{2p+2q} \mapsto \mathbb{R}^q)$ are the algebraic equations, and $h_{\dot{x}}$ is full rank, see also Remark 2.1; $\mathbf{x}$ ($\mathbf{x} \in \mathbb{R}^p$) are the state variables; and $\mathbf{y}$ ($\mathbf{y} \in \mathbb{R}^q$) are the algebraic variables. Here we denote $\mathbf{y}(t-\tau)$ as $\mathbf{y}_d$. Then the above system is called non-index 1 Hessenberg if $\mathbf{y}_d \neq 0$ and $\frac{\partial \mathbf{g}}{\partial \mathbf{y}}$ is singular, i.e. $\det(\frac{\partial \mathbf{g}}{\partial \mathbf{y}}) = 0$.

The NDDEs (1) can be transformed into the comparison set of non-index 1 Hessenberg Delay Differential Algebraic Equations (DDAEs) with equivalent stability characteristic and the same eigenvalues. This numerical appraisal is called *descriptor model transformation* [7]. With this transformation, analysis approaches for DDAEs can be extended to NDDEs. Regarding to the frequency-domain analysis of DDAEs, reference [14] improves the computation efficiency and simplifies the implementation of the eigenvalue-based approach through Chebyshev discretization to obtain the dominant eigenvalues. In [12], the Chebyshev discretization method is shown to achieve the best ratio of accuracy/computational burdens for large-size physical systems, i.e. real-world power systems. The Chebyshev discretization was developed to solve non-index 1 Hessenberg form DDAEs [13]. Reference [20] discusses the basic idea of the eigenvalue analysis of NDDEs based on the transformation to non-index 1 Hessenberg form.

The remainder of the paper is organized as follows. In Section 2 we derive the expression of the characteristic equation of systems of NDDEs, based on the transformation into a non-index 1 Hessenberg form DDAE. Section 2 also proves a Theorem of sufficient and necessary conditions for delay-independent stability of (1) and eigenvalue analysis approach of NDDEs . Section 3 presents several working examples of the numerical appraisals discussed in Section 2. Conclusions are drawn in Section 4.

2 Stability Analysis of Neutral Systems

2.1 Eigenvalue Analysis

This section defines the characteristic equation of (1), considering a single-delay case. The extension to multiple-delay cases is straightforward. To simplify the development of the proofs included in this section, let:

$$\mathbf{x}_d = \mathbf{x}(t-\tau), \quad \mathbf{x}_{2d} = \mathbf{x}(t-2\tau), \quad \ldots \quad \mathbf{x}_{kd} = \mathbf{x}(t-k\tau), \quad \forall k \in \mathbb{N}_+ \ ,$$

be *retarded* or *delayed* state and algebraic variables, respectively, where t is current simulation time, and τ ($\tau > 0$) is time delay. In the remainder of this section, since the main focus is on frequency-domain stability analysis, time delays are assumed to be constant. Based on above expression, (1) can be rewritten as:

$$\mathbf{0}_{p,1} = \mathbf{f}(\mathbf{x}, \mathbf{x}_d, \dot{\mathbf{x}}, \dot{\mathbf{x}}_d) \ .$$

It is worth noticing at this point that we consider only *small disturbances*, e.g., disturbances whose effects on the stability of a given equilibrium point can be studied through the linearized set of the equations that model the system. According to this assumption, we can now state the following stability theorem:

Theorem 2.1. Consider system (1) with full rank $f_{\dot{x}}$ at a equilibrium point. Then following a small disturbance, a necessary and sufficient condition for the equilibrium solution to be asymptotically stable is that the roots of $\Delta(\lambda)$ all have negative real parts, where $\Delta(\lambda)$ is given by:

$$\Delta(\lambda) = \lambda \mathbf{I}_p - \mathbf{A}_0 - e^{-\lambda\tau}\mathbf{A}_1 - \sum_{k=2}^{\infty} e^{-\lambda k\tau}\mathbf{A}_k \ , \tag{3}$$

with:

$$\mathbf{A}_0 = \mathbf{A}, \quad \mathbf{A}_1 = \mathbf{D}, \quad \mathbf{A}_k = \mathbf{C}^{k-1}\mathbf{D}, \quad k \geq 2 \ ,$$

and

$$\mathbf{A} = -\mathbf{f}_{\dot{\mathbf{x}}}^{-1}\mathbf{f}_{\mathbf{x}}, \quad \mathbf{B} = -\mathbf{f}_{\dot{\mathbf{x}}}^{-1}\mathbf{f}_{\mathbf{x}_d}, \quad \mathbf{C} = -\mathbf{f}_{\dot{\mathbf{x}}}^{-1}\mathbf{f}_{\dot{\mathbf{x}}_d}, \quad \mathbf{D} = \mathbf{B} + \mathbf{C}\mathbf{A} \ .$$

In this case the series in (3) converges if $\rho(\mathbf{C}) < 1$, where $\rho(\cdot)$ is the spectral radius of the eigenvalues of a matrix.

Proof. Consider a small disturbances at a equilibrium point, with Δ presents the deviation from this point, we linearize (1):

$$\mathbf{0}_{p,1} = \mathbf{f}_{\mathbf{x}}\Delta\mathbf{x} + \mathbf{f}_{\mathbf{x}_d}\Delta\mathbf{x}_d + \mathbf{f}_{\dot{\mathbf{x}}}\Delta\dot{\mathbf{x}} + \mathbf{f}_{\dot{\mathbf{x}}_d}\Delta\dot{\mathbf{x}}_d \ . \tag{4}$$

The *characteristic equation* of (4) is given by

$$\det \Delta(\lambda) = 0 \ ,$$

where

$$\Delta(\lambda) = \lambda(\mathbf{f}_{\dot{\mathbf{x}}} + e^{-\lambda\tau}\mathbf{f}_{\dot{\mathbf{x}}_d}) + \mathbf{f}_{\mathbf{x}} + e^{-\lambda\tau}\mathbf{f}_{\mathbf{x}_d} \ , \tag{5}$$

is the *characteristic matrix*. The solutions λ of the characteristic equation are called the *characteristic roots*, *spectrum*, or *eigenvalues*. Such eigenvalues allow defining the local stability properties for the stability for nonlinear NDDEs (1) at their equilibria and global stability properties for linear NDDEs.

Instead of solving the above complicated characteristic equation directly, we propose to solve an equivalent equation, which is determined based on a variable transformation of (1). Let $\mathbf{y} = \dot{\mathbf{x}}$

and $\mathbf{f}_{\dot{\mathbf{x}}}$ be full rank, then (1) can be rewritten as:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{y} \\ \mathbf{0}_{p,1} &= \mathbf{f}(\mathbf{x}, \mathbf{x}_d, \mathbf{y}, \mathbf{y}_d) \ , \end{aligned} \tag{6}$$

which is a set of DDAEs (see Definition 1.1). This is a typical descriptor model transformation [7]. Note that if $\dot{\mathbf{x}} \neq \mathbf{0}_{p,1}$ then system (6) is always non-Hessenberg index 1. Differentiating (6) at the equilibrium point leads to:

$$\begin{aligned} \Delta\dot{\mathbf{x}} &= \Delta\mathbf{y} \\ \mathbf{0}_{p,1} &= \mathbf{f}_{\mathbf{x}}\Delta\mathbf{x} + \mathbf{f}_{\mathbf{x}_d}\Delta\mathbf{x}_d + \mathbf{f}_{\mathbf{y}}\Delta\mathbf{y} + \mathbf{f}_{\mathbf{y}_d}\Delta\mathbf{y}_d \ , \end{aligned} \tag{7}$$

where $\mathbf{f}_{\mathbf{y}} \equiv \mathbf{f}_{\dot{\mathbf{x}}}$ and $\mathbf{f}_{\mathbf{y}_d} \equiv \mathbf{f}_{\dot{\mathbf{x}}_d}$. Note that, if $\mathbf{f}_{\mathbf{y}_d} \neq \mathbf{0}_{p,p}$, then (7) is a set of non-index 1 Hessenberg form DDAEs. The derivation of the characteristic equation of general non-index 1 Hessenberg form DDAEs is thoroughly discussed in [13]. Such DDAEs have the following characteristic matrix,see [13]:

$$\Delta(\lambda) = \lambda\mathbf{I}_p - \mathbf{A}_0 - e^{-\lambda\tau}\mathbf{A}_1 - \sum_{k=2}^{\infty} e^{-\lambda k\tau}\mathbf{A}_k \ , \tag{8}$$

where $\mathbf{I}_p$ is the identity matrix of order p, based on the the specific form of (7),

$$\mathbf{A}_0 = \mathbf{A}, \quad \mathbf{A}_1 = \mathbf{D}, \quad \mathbf{A}_k = \mathbf{C}^{k-1}\mathbf{D}, \quad k \geq 2$$

and

$$\mathbf{A} = -\mathbf{f}_{\dot{\mathbf{x}}}^{-1}\mathbf{f}_{\mathbf{x}}, \quad \mathbf{B} = -\mathbf{f}_{\dot{\mathbf{x}}}^{-1}\mathbf{f}_{\mathbf{x}_d}, \quad \mathbf{C} = -\mathbf{f}_{\dot{\mathbf{x}}}^{-1}\mathbf{f}_{\dot{\mathbf{x}}_d}, \quad \mathbf{D} = \mathbf{B} + \mathbf{C}\mathbf{A} \ .$$

Since systems (4), and (7) are equivalent, their characteristic matrices (5), (8) respectively, will be equivalent as well. For asymptotic stable states we have that $\mathrm{Re}(\lambda) < 0$, or, equivalently, since $\tau > 0$, $\tau\mathrm{Re}(\lambda) < 0$. Then

$$|e^{\tau[\mathrm{Re}(\lambda)+i\mathrm{Im}(\lambda)]}| < |e^{i\tau\mathrm{Im}(\lambda)}| \ ,$$

or, equivalently,

$$|e^{\tau\lambda}| < 1 \ .$$

The matrix series in (3) can be written as:

$$\sum_{k=2}^{\infty} e^{-\lambda k\tau}\mathbf{C}^{k-1}\mathbf{D} = \big(\sum_{k=1}^{\infty}[e^{-\lambda(k+1)\tau}\mathbf{C}^{k}]\big)\mathbf{D} \ .$$

Hence the matrix series $\sum_{k=2}^{\infty} e^{-\lambda k\tau}\mathbf{C}^{k-1}\mathbf{D}$ converges if and only if $\sum_{k=1}^{\infty} e^{-\lambda(k+1)\tau}\mathbf{C}^{k}$ converges. By applying the D'Alembert criterion, $\sum_{k=1}^{\infty} e^{-\lambda(k+1)\tau}\mathbf{C}^{k}$ converges if:

$$\lim_{k\to+\infty}\frac{\left\|e^{-\lambda(k+2)\tau}\mathbf{C}^{k+1}\right\|}{\left\|e^{-\lambda(k+1)\tau}\mathbf{C}^{k}\right\|} < 1 \ ,$$

or, equivalently,

$$|e^{-\lambda\tau}|\lim_{k\to+\infty}\frac{\left\|\mathbf{C}^{k+1}\right\|}{\left\|\mathbf{C}^{k}\right\|} < 1 \ ,$$

by using $\|C^{k+1}\| \leq \|C^{k}\|\|C\|$ we get

$$|e^{-\lambda\tau}|\lim_{k\to+\infty}\frac{\left\|\mathbf{C}^{k+1}\right\|}{\left\|\mathbf{C}^{k}\right\|} \leq |e^{-\lambda\tau}|\lim_{k\to+\infty}\frac{\left\|\mathbf{C}^{k}\right\|\left\|\mathbf{C}\right\|}{\left\|\mathbf{C}^{k}\right\|} < 1 \ ,$$

or, equivalently,

$$\left\|\mathbf{C}\right\| < |e^{\lambda\tau}| < 1 \ ,$$

or, equivalently,

$$\left\|\mathbf{C}\right\| < 1 \ .$$

Hence, the matrix series $\sum_{k=2}^{\infty} e^{-\lambda k\tau}\mathbf{C}^{k-1}\mathbf{D}$ in (3) converges if

$$\rho(\mathbf{C}) = \rho(\mathbf{f}_{\mathbf{y}}^{-1}\mathbf{f}_{\mathbf{y}_d}) < 1$$

holds. The proof is completed. □

Remark 2.1. Where $\mathbf{f}_{\mathbf{y}}^{-1}$ certainly exists as $\mathbf{f}_{\mathbf{y}} = \mathbf{f}_{\dot{\mathbf{x}}}$ is assumed to be full rank in this type of problems. However, the assumption that $\mathbf{f}_{\mathbf{y}}$ is full rank does not reduce the generality of the approach proposed in this paper. In fact, if $\mathbf{f}_{\mathbf{y}}$ has rank q, $q < p$, (4) can be always rewritten as a set of DDAEs for which the Jacobian matrix $\tilde{\mathbf{f}}_{\tilde{\mathbf{y}}}$ with respect of a subset of the state variables $\tilde{\mathbf{x}} \in \mathbf{x}$ is full rank.

Remark 2.2. If $\mathbf{f}_{\mathbf{y}}$ is full rank, (4) can be rewritten in an explicit form by multiplying by $-\mathbf{f}_{\mathbf{y}}^{-1}$:

$$\begin{aligned} \Delta\dot{\mathbf{x}} &= \Delta\mathbf{y} \\ \mathbf{0}_{p,1} &= \hat{\mathbf{f}}_{\mathbf{x}}\Delta\mathbf{x} + \hat{\mathbf{f}}_{\mathbf{x}_d}\Delta\mathbf{x}_d - \Delta\mathbf{y} + \hat{\mathbf{f}}_{\mathbf{y}_d}\Delta\mathbf{y}_d \ , \end{aligned}$$

and hence:

$$\mathbf{A} = \hat{\mathbf{f}}_{\mathbf{x}}, \quad \mathbf{B} = \hat{\mathbf{f}}_{\mathbf{x}_d}, \quad \mathbf{C} = \hat{\mathbf{f}}_{\mathbf{y}_d}, \quad \mathbf{D} = \hat{\mathbf{f}}_{\mathbf{x}_d} + \hat{\mathbf{f}}_{\mathbf{y}_d}\hat{\mathbf{f}}_{\mathbf{x}}.$$

If $\rho(\mathbf{C}) < 1$, the matrices $\mathbf{A}_k$ tend to $\mathbf{0}_{p,p}$ as $k \to \infty$. Based on the definition of $\mathbf{A}_k$, the condition $\rho(\mathbf{f}_{\mathbf{y}}^{-1}\mathbf{f}_{\mathbf{y}_d}) < 1$ must hold, which, by using the above explicit formulation becomes $\rho(\hat{\mathbf{f}}_{\mathbf{y}_d}) < 1$.

Remark 2.3. Equation (3) includes a series of infinite terms, which, in actual implementations, has to be truncated at a given value of k_m (see [13]). In the examples given in the following section, we thus approximate (3) as:

$$\Delta(\lambda) = \lambda\mathbf{I}_p - \mathbf{A}_0 - e^{-\lambda\tau}\mathbf{A}_1 - \sum_{k=2}^{k_m} e^{-\lambda k\tau}\mathbf{A}_k \ , \tag{9}$$

where k_m has to be large enough.

The Chebyshev discretization scheme approach solves the eigenvalues through transforming the original problem of computing the roots of (9) into a matrix eigenvalue problem of a Partial Differential Equation's (PDE) system of infinite dimensions. The dimension of the PDE is made tractable using a discretization based on a finite element method. The discretized matrix is build as follows. Let Ξ_N be the Chebyshev discretization matrix of order N (see [14] for details) and define

$$\mathbf{M} = \left[\begin{array}{ccccc} & & \hat{\mathbf{\Psi}} \otimes \mathbf{I}_p & & \\ \hline \hat{\boldsymbol{A}}_N & \hat{\boldsymbol{A}}_{N-1} & \dots & \hat{\boldsymbol{A}}_1 & \hat{\boldsymbol{A}}_0 \end{array} \right] ,$$

where $\otimes$ indicates the *tensor product* or the Kronecker product; $\hat{\mathbf{\Psi}}$ is a matrix composed of the first $N-1$ rows of $\mathbf{\Psi}$ defined as

$$\mathbf{\Psi} = -2\mathbf{\Xi}_N/\tau ,$$

and the matrices $\hat{\boldsymbol{A}}_0, \dots, \hat{\boldsymbol{A}}_N$ are defined as follows:

Equation (9) has k_m delays, with $\tau = \tau_1 < \tau_2 < \dots < \tau_{k_m-1} < \tau_{k_m} = k_m\tau$. Each point of the Chebyshev grid corresponds to a delay $\theta_j = (N-j)\Delta\tau$, with $j = 1, 2, \dots, N$ and $\Delta\tau = \tau_{k_m}/(N-1)$. Thus, $j = 1$ corresponds to the state matrix $\mathbf{A}_{k_m}$, which corresponds to the maximum delay τ_{k_m}; and $j = N$ is taken by the non-delayed state matrix $\mathbf{A}_0$. If a delay $\tau_k = \theta_j$ for some $j = 2, \dots, N-1$, then the correspondent matrix $\mathbf{A}_k$ takes the position j in the grid. The delays in (9) are equally spaced and, hence, these conditions happen if N is a multiple of k_m. The linear interpolation discussed in [14] allows a easily extension to the multi-delay case. The number of points N of the grid affects the precision and the computational burden of the method, as it is discussed in [20].

2.2 Delay-independent Stable Criterion

From the point of the frequency-domain approach, we can extend the delay-independent criterion of a linear retarded system [17] to neutral system based on the deduced Characteristic equation in the last section.

Definition 2.1. *Delay margin* is a constant value τ_c, that $\forall \tau \in [0, \tau_c]$ in a system of (1) is stable. If τ_c is infinity, the system is delay-independent stable.

Definition 2.2. A square matrix $\mathbf{A}$ is called stable matrix (or Hurwitz matrix) if every eigenvalue of A has strictly negative real part.

Theorem 2.2. We consider the NDDE system (1) and its characteristic equation (3). If $\rho(\mathbf{C}) < 1$, then the following sufficient and necessary conditions must hold for delay - dependent stability of (1):

(i) $\mathbf{A}_0$ is stable;

(ii) $\mathbf{N}_{km} = \mathbf{A}_0 + \mathbf{A}_1 + \sum_{k=2}^{k_m} \mathbf{A}_k$ is stable;

(iii) $\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega)) > 1, \forall \omega \in (0, \infty)$, where

$$\underline{\rho}(\mathbf{G}(s), \mathbf{H}(s)) = \frac{1}{\rho(\mathbf{G}^{-1}(s)\mathbf{H}(s))}, \quad \mathbf{H}(s) = \mathrm{diag}(\mathbf{I}_p \ \dots \ \mathbf{I}_p \ -\mathbf{A}_{km})$$

and

$$\mathbf{G}(s) = \begin{bmatrix} \mathbf{0}_p & \mathbf{I}_p & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_p & \mathbf{0}_p & \dots & \mathbf{I}_p \\ -(s\mathbf{I}_p - \mathbf{A}_0) & \mathbf{A}_1 & \dots & \mathbf{A}_{km-1} \end{bmatrix}.$$

Proof. If $\tau \to \infty$ then (3) takes the form

$$\Delta(\lambda) = \lambda \mathbf{I}_p - \mathbf{A}_0.$$

Hence, the matrix $\mathbf{A}_0$ has to be stable in order to have stability for (1) at the equilibrium state. If $\tau = 0$ then (9) takes the form

$$\Delta(\lambda) = \lambda \mathbf{I}_p - \mathbf{A}_0 - \mathbf{A}_1 - ... - \mathbf{A}_{km},$$

which means that the matrix $\mathbf{A}_0 + \mathbf{A}_1 + \sum_{k=2}^{k_m} \mathbf{A}_k$ has to be stable in order to have stability for (1) at the equilibrium state. Equation (9)

$$\Delta(\lambda) = \lambda \mathbf{I}_p - \mathbf{A}_0 - e^{-\lambda\tau}\mathbf{A}_1 - \sum_{k=2}^{k_m} e^{-\lambda k\tau}\mathbf{A}_k,$$

is also the characteristic equation of the matrix differential equation

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \mathbf{A}_1\mathbf{x}_d + \mathbf{A}_1\mathbf{x}_{2d} + ... + \mathbf{A}_{k_m}\mathbf{x}_{k_m d}.$$

By applying the Fourier transform to the above equation, $\mathcal{F}(\mathbf{x}) = \mathbf{X}(\omega)$, we get

$$j\omega\mathbf{X}(\omega) = \mathbf{A}_0\mathbf{X}(\omega) + \mathbf{A}_1 e^{-j\omega\tau}\mathbf{X}(\omega) + \mathbf{A}_1 e^{-j\omega 2\tau}\mathbf{X}(\omega) + ... + \mathbf{A}_{k_m} e^{-j\omega k_m \tau}\mathbf{X}(\omega),$$

or, equivalently,

$$\mathbf{X}(\omega) = \sum_{k=1}^{k_m} (j\omega\mathbf{I}_p - \mathbf{A}_0)^{-1}\mathbf{A}_k e^{-j\omega k\tau}\mathbf{X}(\omega).$$

Similar to the fixed-point iteration method, we may argue that the above equation is written in such a way that any of its solution, which is a fixed point of $\sum_{k=1}^{k_m} (j\omega\mathbf{I}_p - \mathbf{A}_0)^{-1}\mathbf{A}_k e^{-j\omega k\tau}\mathbf{X}(\omega)$, is also a solution of the comparison equation. Then we may consider the following algorithm. If we start from any point and consider the recursive process

$$\mathbf{x}_n = \sum_{k=1}^{k_m} (j\omega\mathbf{I}_p - \mathbf{A}_0)^{-1}\mathbf{A}_k \mathbf{x}_{n-k},$$

then if $\mathbf{x}_n$ converges there exist a solution for the equation that arises after the Fourier transform and hence the system of differential equations is stable, i.e., the roots of its characteristic equation have strictly negative real part and we conclude that we have delay-dependent stability for (1). Hence we have to derive the condition under which the solution of the above matrix difference equation converges. We adopt the following notation:

$$\mathbf{y}_n^{(1)} = \mathbf{x}_{n-1}, \;\; \mathbf{y}_n^{(2)} = \mathbf{x}_{n-2}, \;\; \dots \;\; \mathbf{y}_n^{(k_m-1)} = \mathbf{x}_{n-(k_m-1)}, \;\; \mathbf{y}_n^{(k_m)} = \mathbf{x}_{n-k_m},$$

or, equivalently,

$$\mathbf{y}_{n+1}^{(1)} = \mathbf{x}_n, \;\; \mathbf{y}_{n+1}^{(2)} = \mathbf{x}_{n-1}, \;\; \dots \;\; \mathbf{y}_{n+1}^{(k_m-1)} = \mathbf{x}_{n-(k_m-2)}, \;\; \mathbf{y}_{n+1}^{(k_m)} = \mathbf{x}_{n-(k_m-1)}.$$

Furthermore,

$$\begin{gathered}
\mathbf{y}_{n+1}^{(1)} = \textstyle\sum_{k=1}^{k_m} (j\omega I_p - \mathbf{A}_0)^{-1}\mathbf{A}_k\mathbf{y}_k^{(k)}, \\
\mathbf{y}_{n+1}^{(2)} = \mathbf{y}_n^{(1)}, \\
\vdots \\
\mathbf{y}_{n+1}^{(k_m-1)} = \mathbf{y}_n^{(k_m-2)}, \\
\mathbf{y}_{n+1}^{(k_m)} = \mathbf{y}_n^{(k_m-1)}.
\end{gathered}$$

Or, in matrix form

$$\mathbf{Y}_{n+1} = \mathbf{M}_{k_m}(j\omega)\mathbf{Y}_n,$$

where

$$\mathbf{Y}_n = \begin{bmatrix} \mathbf{y}_n^{(1)} \\ \mathbf{y}_n^{(2)} \\ \dots \\ \mathbf{y}_n^{(k_m)} \end{bmatrix}$$

and

$$\mathbf{M}_{k_m}(j\omega) = \begin{bmatrix} (j\omega\mathbf{I}_p - \mathbf{A}_0)^{-1}\mathbf{A}_1 & \dots & (j\omega\mathbf{I}_p - \mathbf{A}_0)^{-1}\mathbf{A}_{k_m-1} & (j\omega\mathbf{I}_p - \mathbf{A}_0)^{-1}\mathbf{A}_{k_m} \\ \mathbf{I}_p & \dots & \mathbf{0}_{p,p} & \mathbf{0}_{p,p} \\ \mathbf{0}_{p,p} & \dots & \mathbf{0}_{p,p} & \mathbf{0}_{p,p} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{p,p} & \dots & \mathbf{I}_p & \mathbf{0}_{p,p} \end{bmatrix}.$$

Hence in order $\mathbf{x}_n$ to converge, $\mathbf{Y}_n$ has to converge, i.e.,

$$\rho(\mathbf{M}_{k_m}(j\omega)) < 1.$$

From [17] it can be easily observed that $\mathbf{M}_{k_m}(j\omega) = \mathbf{G}^{-1}(j\omega)\mathbf{H}(j\omega)$ and thus

$$\rho(\mathbf{G}^{-1}(j\omega)\mathbf{H}(j\omega)) < 1,$$

or, equivalently,

$$\frac{1}{\rho(\mathbf{G}^{-1}(j\omega)\mathbf{H}(j\omega))} > 1,$$

or, equivalently,

$$\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega)) > 1.$$

The proof is completed. □

3 Case Study

This section further provides numerical appraisals of the conceptions discussed in Section 2 through three NDDE systems written in the form of (1). The objectives of each case are listed below.

1. Partial Element Equivalent Circuit case in Section 3.1 explains the applications of Theorem 2.2 and proves the severe conservativeness of traditional LFMs. This section also points out the limitation of the transforming approach discussed in Remark 2.3.

2. Dynamic population model in Section 3.2.1 is an example that shows how Theorem 2.2 asserting delay-dependent stability on a nonlinear NDDE.

3. Neutral system with distributed delay in Section 3.2.2 further proves the accuracy of Remark 2.3 and extends its application to non-constant delay case.

In this section, the roots of (9) are obtained by using the Python Package DOME [11]. All simulations and computations in this Section have been executed on a 64-bit Linux Fedora 21 operating system running on a two Intel Xeon 10 Core 2.2 GHz CPUs, 64 GB of RAM, and a 64-bit NVidia Tesla K20X GPU.

3.1 Delay-independent neutral systems

Reference [22] discusses a third-order Partial Element Equivalent Circuit (PEEC), which is described as following NDDE:

$$\dot{\mathbf{x}}(t) = \mathbf{L}\mathbf{x}(t) + \mathbf{M}\mathbf{x}(t-\tau_m) + \mathbf{N}\dot{\mathbf{x}}(t-\tau_m) \ , \tag{10}$$

where,

$$\frac{\mathbf{L}}{100} = \begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \frac{\mathbf{M}}{100} = \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix} \text{ and } \mathbf{N} = \frac{1}{72}\begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix} .$$

Reference [22] proves the zero solution of this system is asymptotic stable if τ_m is *small* but leaves the exact delay margin of the system as an open question. Then, reference [4, 6, 15, 21] obtain conservative delay margins as 0.43 s, 1.1413 s, 1.5022 s and 1.6851 s respectively, through LFMs.

The PEEC system satisfied the hypotheses of the Theorem 2.2 because $\rho(\mathbf{C}) = \rho(\mathbf{N}) = 0.0733 \ll 1$. The $\mathbf{A}_0(\mathbf{L})$ has a very negative rightmost eigenvalues, namely -407.93. Since $\mathbf{A}_0$ is stable, the condition Theorem 2.2-i holds. For $k_m \geqslant 20$, the real part of rightmost eigenvalues of $\mathbf{N}_{k_m}$ converges to -0.0527, the condition Theorem 2.2-ii, thus, holds. The proof of condition Theorem 2.2-iii is computationally intensive. The convergence of $\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega))$ with the increase of k_m is slow. Figure 1-a shows the values of $\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega))|_{\omega=0}$ as a function of $k_m \in [5, 500]$. Figure 1-a indicates that the function $\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega))$ has a proper convergence and no rounding error for $k_m = 350$. To further study the condition iii, Figure 1-b, then, shows that the values of $\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega))$ for $\omega \in [1, 200]$ with $k_m = 350$, which are within the range $[1.122, 1.127]$. According to Figure 1 and the continuousness of the function, $\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega)) > 1$, $\forall \omega \in (0, \infty)$ will be held. This model, therefore, satisfies the delay-independent criterion Theorem 2.2-iii. Above results prove the system is delay-independent stable at zero solution yield. Since the PEEC system is linear, it is globally delay-independent stable.

Then, we also study the network through the Chebyshev discretization approach. Figure 2 shows the real part of estimated rightmost eigenvalues of the PEEC model with unbounded constant delay τ. As τ tends to infinity, the estimated rightmost eigenvalues go to zero, which, however, are supposed to converge to the eigenvalues of matrix $\mathbf{A}_0$ (see Section 2), whose rightmost eigenvalue is -407.93. The spurious zero eigenvalues are introduced by the Chebyshev discretization matrix $\boldsymbol{\Psi} = -2\Xi_N/\tau$ when τ goes to infinity. Although the Chebyshev discretization approach may fail to capture the precise dominating eigenvalues for large delays, it always provides accurate stability assertion. We will show its accuracy to find delay margins in the following section.

3.2 Delay-dependent neutral systems

This section focuses on obtaining the exact delay margin of the delay-dependent stable neutral systems through the numerical approach discussed in Section 2.

3.2.1 Dynamic food-limited population model

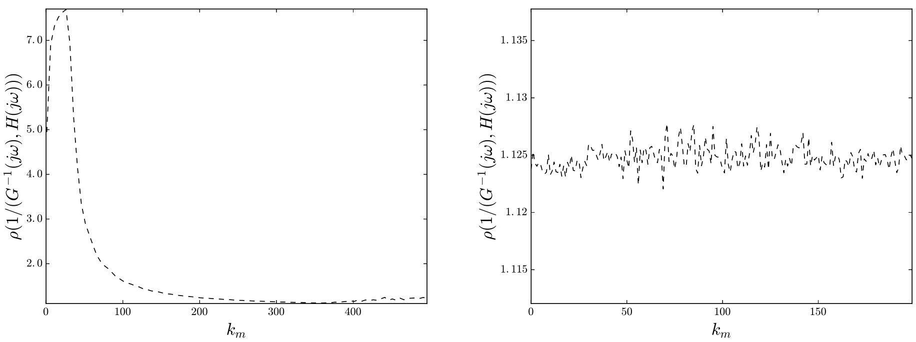
The dynamic food-limited population model introduced in [16] is a typical nonlinear system of NDDE in the form of (1):

$$\dot{S}(t) = rS(t)\left[1 - \frac{S(t-\tau) + c\dot{S}(t-\tau)}{K}\right], \tag{11}$$

where r and τ are intrinsic growth rate and the recovering time, respectively, of species S, and K is the environment capacity. Parameters r, c, K are positive.

Reference [10] provides a numerical example of the dynamic bacteria population model (11), with $K = 1$, and

$$r = \frac{\pi}{\sqrt{3}} + \frac{1}{20}, \quad \text{and} \quad c = \frac{\sqrt{3}}{2\pi} - \frac{1}{25}.$$



(a) $\underline{\rho}(\boldsymbol{G}(j\omega), \boldsymbol{H}(j\omega))|_{\omega=0} \propto k_m$ (b) $\underline{\rho}(\boldsymbol{G}(j\omega), \boldsymbol{H}(j\omega))|_{k_m=350} \propto \omega$.

Figure 1: $\underline{\rho}(\boldsymbol{G}(j\omega), \boldsymbol{H}(j\omega))$ *of the PEEC* (10).

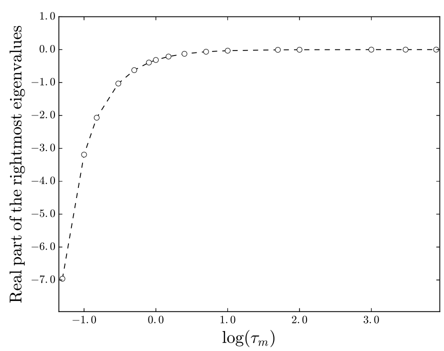


Figure 2: Rightmost eigenvalues of the PEEC (10)*as a function of* τ_m *and with* $k_m = 350$ *and* $N = 300$.

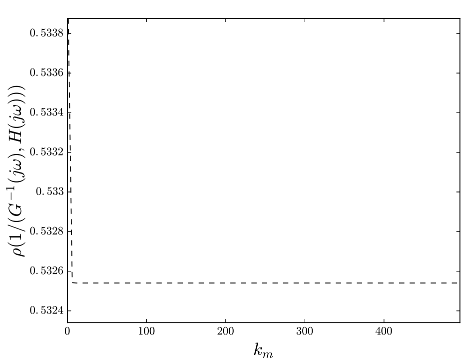


Figure 3: $\underline{\rho}(\boldsymbol{G}(j\omega), \boldsymbol{H}(j\omega))|_{\omega=1}$ *of the dynamic population model (11) as a function of* k_m

Linearize the model at the equilibrium point $S(t) = K$:

$$\dot{S}(t) = -1.81S(t-\tau) - 0.5\dot{S}(t-\tau) \ , \tag{12}$$

According to above linearization, it is easy to show the system satisfies the conditions i and ii of Theorem i and ii. The trajectory of $\underline{\rho}(\mathbf{G}(j\omega), \mathbf{H}(j\omega))|_{\omega=1}$ as a function of $k_m \in [5, 500]$ is shown in Figure 3. The function converges at 0.5325 as k_m increases. This result violates the condition iii. Therefore, this dynamic population model is delay-dependent stable at the stationary point $S = K$, according to Theorem 2.2-iii. Reference [20] shows the delay margin is approximately 14.7 s.

3.2.2 Delay-dependent neutral systems with distributed delay

In this section, we consider a well-discussed neutral delay system with inclusion of a distributed delay.

$$\dot{\mathbf{x}}(t) - \mathbf{C}\dot{\mathbf{x}}(t-\tau) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-h) + \mathbf{D}\int_{t-r}^{t} \mathbf{x}(s)ds \ . \tag{13}$$

where:

$$\mathbf{A} = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -0.12 & -0.12 \\ -0.12 & -0.12 \end{bmatrix}.$$

The distributed delay of (13) can be transfromed into a sum of multiple constant delays through the Simpson Second rule [9] with the subinterval number equal to 15.

The delay margins of the following two scenarios are studied:

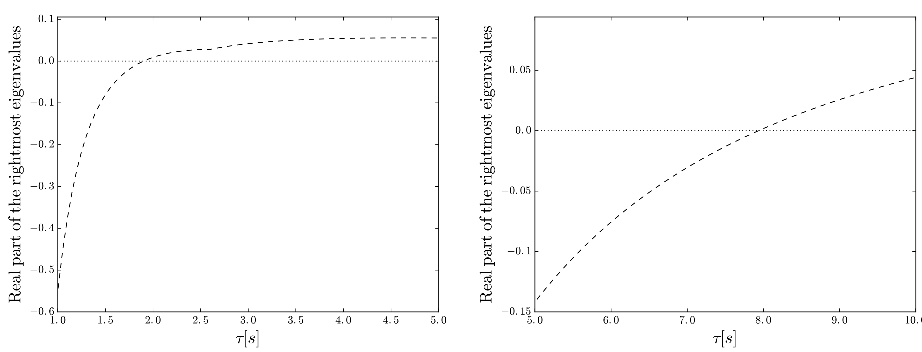
a. $h = 0.1$ s, $r = 0.1$ s. The evolution of the real part of the rightmost eigenvalues as a function of τ is shown in Figure 4-a.

b. $h = 1$ s, $\tau = 1$ s. The evolution of the real part of the rightmost eigenvalues as a function of r is shown in Figure 4-b.

The delay margins of above two scenarios are easy to find from Figure 4. The same scenarios have also been discussed in several other papers. Table 1 compares the delay margins of the two scenarios obtained through LFMs. The first row of Table 1 indicates the reference that provides each result. As shown in Table 1, the LFMs [25, 28, 30] provide conservative delay margins of the neutral system (13). Although the numerical approach provided by [5] deduces the most accurate delay margin in scenario *a*, its result for scenario *b* is too aggressive and questionable.

Table 1: Delay Margins of NDDE (13)

Scenario	[28]	[25]	[30]	[5]	This paper
$a.$	1.1 s	1.2 s	1.3 s	1.9 s	1.9 s
$b.$	6.2 s	6.4 s	6.6 s	>100 s	7.9 s



(a) *(b)*

Figure 4: Rightmost eigenvalue of (13) scenarios as a function of τ *and with* $N = 300$*,* $N_t = 15$ *and* $k_m = 100$*.*

4 Conclusions

The paper provides a derivation of the characteristic equation of a class of systems described by NDDEs and a systematic approach to solve the stability analysis of such NDDEs systems. The characteristic equation is found by means of a descriptor model transformation that leads to an equivalent non-index 1 Hessenberg form DDAE. The equivalent DDAE characteristic equation consists of a series of terms corresponding to infinitely many delays that are multiples of the delays of the original NDDE. The proposed approach includes two parts, namely, (i) judging whether the neutral system is delay-independent stable; and (ii) solving the eigenvalues for the delay-dependent stable system. Case studies indicate that the proposed method allows determining precisely the delay stability margin and, at least for the considered cases, it allows improving the results obtained with other methods that are available in the literature.

Acknowledgments

This material is supported by the Science Foundation Ireland, by funding Muyang Liu and Federico Milano, under Investigator Programme Grant No. SFI/15/IA/3074 and Ioannis Dassios, under Grant No. SFI/09/SRC/E1780.

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