# Stability Criterion of a Class of Non-Causal Systems of Differential Equations

Ioannis Dassios[1*], Georgios Tzounas[2], Federico Milano[1]

[1]FRESLIPS, University College Dublin, Ireland

[2]ETH Zürich, Switzerland

[*]Corresponding author

**Abstract:** In this paper we focus on a class of non-causal systems of differential equations, namely systems the variables of which can depend not only from the current or past time, but also from future time. For this type of systems, we study their solutions and present new and easily testable conditions under which any state of the system is stable. The stability analysis of a future-state-dependent set of differential equations has its relevance also in practical applications. Numerical examples, as well as an application in electric power engineering, are provided to justify our theory.

**Keywords** : differential equations, non-causality, time-advance, stability.

## 1 Introduction

A system is *causal* if its behavior is dependent upon information from the past and the present, but not from the future. The dynamic response and stability of causal systems is studied using well-established mathematical tools from the theory of functional differential equations, e.g. see [10]. The requirement of causality in the definition of dynamical systems [1, 19], neutral dynamical systems [6, 15], and for example their applications in Electrical Power Systems [12, 14, 16, 20] & Macroeconomics [7], has been a long-standing system property .

This paper focuses on the study of a class of systems that violate the system property of causality, i.e. systems whose behavior is impacted by some future or *time-advanced* information. Such systems, referred as *non-causal* or *anticipatory* [18], are difficult, if not impossible, to realize for real-time operation [5], and thus their identification, estimation, and control are challenging problems to address. Note here that the presence of time-advanced information in a system does not necessarily imply the violation of causality, e.g. see the discussions on negative group delays and causality in electronic circuits, in [11, 17].

Non-causality is relevant in a number of existing applications. A couple of examples are the output tracking control of non-minimum phase systems [8], and non-causal filtering [2, 9, 21]. The latter, although not evidently realizable for real-time processing, is widely employed in signal processing, in particular for offline analysis of historical time-series.

The scientific community that studies the modelling and numerical integration of physical systems, often defines a *non-causal* system as a system the equations of which are written in such a way that there is no *cause-effect* among variables as opposed to, for example, control systems, where one has inputs and outputs. Note though that this definition of causality is not what we discuss in this article. For the sake of completeness, we mention that an interesting discussion on *causality* for the implementation of physical systems with the meaning above can be found in [3].

In many control applications it is relevant to utilize the time derivative of controlled quantities as is, for example, the case of Proportional Integral Derivative (PID) controllers. Since, the time derivative of a signal provides an indication on the trend in the signal, it can be utilized to estimate the future value of the controlled signal itself. A relevant application is the compensation of delayed quantities, see for example [4, 13, 22]. In fact, an ideal compensation would be able to estimate exactly the value of the signal in the future. In such application, the study of the stability of future-state-dependent set of differential equations can provide the limit of the stability of controls that are based on compensation.

The remainder of the article is organized as follows. Section 2 describes the formulation of the non-causal system under study and provides preliminary results on its solution. The main results of the paper are presented in Section 3. Numerical examples are discussed in Section 4. Finally, conclusions are drawn in Section 5.

# 2 Preliminaries

## 2.1 System Formulation

We consider a dynamical system the evolution of which depends on the value of a *time-advanced* state, $t_{adv} \in [0, \infty)$, i.e. the following set of explicit non-causal differential equations:

$$\boldsymbol{y}'(t) = \boldsymbol{f}\big(\boldsymbol{y}(t), \boldsymbol{y}(t + t_{adv})\big) \,, \tag{1}$$

where $t \in [0, \infty)$; $\boldsymbol{y}(t) \in \mathbb{C}^m$ is the column vector of state variables, with $\boldsymbol{y}'(t)$ we denote the first order derivative of $\boldsymbol{y}(t)$, and $\boldsymbol{f} : \mathbb{R}^{2m} \to \mathbb{R}^m$ is a set of non-linear functions. Equivalently if we set $\boldsymbol{y} = \boldsymbol{y}(t)$, $\boldsymbol{y}_\mu = \boldsymbol{y}(t + t_{adv})$, then (1) can be written as:

$$\boldsymbol{y}' = \boldsymbol{f}\big(\boldsymbol{y}, \boldsymbol{y}_\mu\big).$$

Let $\boldsymbol{y}_0$ be a steady state solution of (1). Then, considering sufficiently small disturbances, and for the purpose of analysis, (1) can be linearized at $\boldsymbol{y}_0$, as follows:

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{y}_0) + \boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{y}_0, \boldsymbol{y}_0)(\boldsymbol{y} - \boldsymbol{y}_0) + \boldsymbol{f}_{\boldsymbol{y}_\mu}(\boldsymbol{y}_0, \boldsymbol{y}_0)(\boldsymbol{y}_\mu - \boldsymbol{y}_0).$$

If we set $\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{y}_0, \boldsymbol{y}_0) = \boldsymbol{f}_{\boldsymbol{y}}$, $\boldsymbol{f}_{\boldsymbol{y}_\mu}(\boldsymbol{y}_0, \boldsymbol{y}_0) = \boldsymbol{f}_{\boldsymbol{y}_\mu}$, $\Delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{y}_0$, $\Delta\boldsymbol{y}_\mu = \boldsymbol{y}_\mu - \boldsymbol{y}_0$, we get:

$$\Delta\boldsymbol{y}' = \boldsymbol{f}_{\boldsymbol{y}}\Delta\boldsymbol{y} + \boldsymbol{f}_{\boldsymbol{y}_\mu}\Delta\boldsymbol{y}_\mu,$$

or, equivalently,

$$\Delta\boldsymbol{y}'(t) = \boldsymbol{f}_{\boldsymbol{y}}\Delta\boldsymbol{y}(t) + \boldsymbol{f}_{\boldsymbol{y}_\mu}\Delta\boldsymbol{y}(t + t_{adv})\,, \tag{2}$$

where $\boldsymbol{f}_{\boldsymbol{y}}$, $\boldsymbol{f}_{\boldsymbol{y}_\mu} \in \mathbb{C}^{m\times m}$, are the Jacobian matrices that correspond to the present-time and time-advanced variables, respectively. Adopting the notation $\mathbf{y} = \Delta\boldsymbol{y}$, $\mathbf{P} = -\boldsymbol{f}_{\boldsymbol{y}}$, $\mathbf{Q} = -\boldsymbol{f}_{\boldsymbol{y}_\mu}$, system (2) can be rewritten in the following form:

$$\mathbf{y}'(t) + \mathbf{P}\mathbf{y}(t) + \mathbf{Q}\mathbf{y}(t + t_{adv}) = \mathbf{0}_{m,1}\,, \tag{3}$$

where $\mathbf{P}, \mathbf{Q} \in \mathbb{C}^{m\times m}$, $\mathbf{y} : [0, +\infty) \to \mathbb{C}^m$, and $t, t_{adv} \geq 0$; and $\mathbf{0}_{m,1}$ denotes the zero matrix of dimensions $m \times 1$.

In this paper we provide new results on the solution and stability of (3).

### 2.2 Preliminary Results

In this subsection we present some preliminary results on system (3).

**Lemma 2.1.** System (3) can be written in the form:

$$\mathbf{y}'(t) + [\mathbf{P} + \mathbf{Q}]\mathbf{y}(t) = \boldsymbol{g}\big(\mathbf{y}, t\big)\,, \tag{4}$$

where $\boldsymbol{g}(\mathbf{y}, t) = \mathbf{Q}\int_t^{t+t_{adv}}[\mathbf{P}\mathbf{y}(u) + \mathbf{Q}\mathbf{y}(u + t_{adv})]du$.

*Proof.* System (3) can be written as:

$$\mathbf{y}'(t) + \mathbf{P}\mathbf{y}(t) + \mathbf{Q}\mathbf{y}(t) - \mathbf{Q}\mathbf{y}(t) + \mathbf{Q}\mathbf{y}(t + t_{adv}) = \mathbf{0}_{m,1}\,,$$

or, equivalently,

$$\mathbf{y}'(t) + [\mathbf{P} + \mathbf{Q}]\mathbf{y}(t) = -\mathbf{Q}[\mathbf{y}(t + t_{adv}) - \mathbf{y}(t)]\,, \tag{5}$$

where

$$-\mathbf{Q}[\mathbf{y}(t + t_{adv}) - \mathbf{y}(t)] = -\mathbf{Q}\int_t^{t+t_{adv}} [\mathbf{y}'(u)]du,$$

and, equivalently,

$$-\mathbf{Q}[\mathbf{y}(t + t_{adv}) - \mathbf{y}(t)] = \mathbf{Q}\int_t^{t+t_{adv}} [\mathbf{P}\mathbf{y}(u) + \mathbf{Q}\mathbf{y}(u + t_{adv})]du\,.$$

Consequently, (5) becomes:

$$\mathbf{y}'(t) + [\mathbf{P} + \mathbf{Q}]\mathbf{y}(t) = \mathbf{Q}\int_t^{t+t_{adv}} [\mathbf{P}\mathbf{y}(u) + \mathbf{Q}\mathbf{y}(u + t_{adv})]du\,,$$

whereby setting $\boldsymbol{g}(\mathbf{y}, t) = \mathbf{Q}\int_t^{t+t_{adv}} [\mathbf{P}\mathbf{y}(u) + \mathbf{Q}\mathbf{y}(u + t_{adv})]du$, we arrive at (4). The proof is completed.

Next we provide a solution to system (3) by using the system formulation in Lemma 2.1.

**Proposition 2.1.** An implicit solution of (3) is given by

$$\mathbf{y}(t) = \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\mathbf{y}(0) + \int_0^t \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}](t-h)}\boldsymbol{g}(\mathbf{y}, h)dh\,, \tag{6}$$

where $\boldsymbol{g}(\mathbf{y}, h) = \mathbf{Q}\int_h^{h+t_{adv}} [\mathbf{P}\mathbf{y}(u) + \mathbf{Q}\mathbf{y}(u + t_{adv})]du$.

*Proof.* From Lemma 2.1, system (3) can be written as:

$$\mathbf{y}'(t) + [\mathbf{P} + \mathbf{Q}]\mathbf{y}(t) = \boldsymbol{g}\big(\mathbf{y}, t\big)\,,$$

with solution in the form:

$$\mathbf{y}(t) = \mathbf{y}_h(t) + \mathbf{y}_p(t)\,, \tag{7}$$

where $\mathbf{y}_h(t)$ is the solution of the system

$$\mathbf{y}'(t) + [\mathbf{P} + \mathbf{Q}]\mathbf{y}(t) = \mathbf{0}_{m,1},$$

and $\mathbf{y}_p(t)$ is a partial solution. Hence:

$$\mathbf{y}_h(t) = \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\mathbf{y}(0)\,, \tag{8}$$

and consequently

$$\mathbf{y}_p(t) = \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\boldsymbol{k}(t)\,, \tag{9}$$

with $\boldsymbol{k}(t)$ being unknown. Substitution of $\mathbf{y}_p(t)$ in (4) yields:

$$\mathbf{y}'_p(t) + [\mathbf{P}+\mathbf{Q}]\mathbf{y}_p(t) = \boldsymbol{g}\big(\mathbf{y}, t\big)\,,$$

or, equivalently,

$$\mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\boldsymbol{k}'(t) = \boldsymbol{g}\big(\mathbf{y}, t\big)\,,$$

or, equivalently,

$$\boldsymbol{k}'(t) = \mathbf{e}^{[\mathbf{P}+\mathbf{Q}]t}\boldsymbol{g}\big(\mathbf{y}, t\big)\,,$$

or, equivalently,

$$\boldsymbol{k}(t) = \int_0^t \mathbf{e}^{[\mathbf{P}+\mathbf{Q}]h}\boldsymbol{g}\big(\mathbf{y}, h\big)dh\,. \tag{10}$$

Substitution of (10) in (9) yields:

$$\mathbf{y}_p(t) = \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\int_0^t \mathbf{e}^{[\mathbf{P}+\mathbf{Q}]h}\boldsymbol{g}\big(\mathbf{y}, h\big)dh\,,$$

or, equivalently,

$$\mathbf{y}_p(t) = \int_0^t \mathbf{e}^{[\mathbf{P}+\mathbf{Q}](h-t)}\boldsymbol{g}\big(\mathbf{y}, h\big)dh\,. \tag{11}$$

By using (11) and (8) in (5), we arrive at (6). The proof is completed.

We close this section with the following two definitions that will be used in the next section.

**Definition 2.1.** A continuous column of functions $\mathbf{y}$ on $[0, +\infty)$ is bounded if $\forall\epsilon > 0$, $\exists\delta > 0$ such that $\|\mathbf{y}\| < \frac{\epsilon}{2}$, $\forall t \geq \delta$.

**Definition 2.2.** Let $\|\cdot\|$ be an induced norm, and $\mathbf{y}$ a bounded continuous column of functions in $[0, +\infty)$. Then an operator $\mathcal{T}$ is called a contraction operator on $\mathbf{y}$ if there exists $q \in [0, 1)$ such that:

$$\|\mathcal{T}\mathbf{y} - \mathcal{T}\mathbf{w}\| \leq q\|\mathbf{y} - \mathbf{w}\|\,,$$

for all $\mathbf{y}$, $\mathbf{w}$ bounded continuous columns of functions on $[0, +\infty)$. Where $\|\mathbf{y} - \mathbf{w}\| = \sup_{t\geq 0}\|\mathbf{y}(t) - \mathbf{w}(t)\|$.

# 3 Stability Results

This section presents the main results of the paper on system (3). We first provide the following definition:

**Definition 3.1.** We define the operator $\mathcal{T}$ on the set of the bounded continuous columns of functions $\mathbf{y}$ on $[0, +\infty)$ as:

$$\mathcal{T}\mathbf{y}(t) = \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\mathbf{y}(0) + \int_0^t \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}](t-h)}\boldsymbol{g}(\mathbf{y}, h)dh\,,$$

where $\boldsymbol{g}(\mathbf{y}, h) = \mathbf{Q}\int_h^{h+t_{adv}}[\mathbf{P}\mathbf{y}(u) + \mathbf{Q}\mathbf{y}(u + t_{adv})]du$.

The following theorem provides sufficient conditions for the stability of (3):

**Theorem 3.1.** Any steady state of system (3) is stable if the following conditions hold:

1. The matrix $\mathbf{P} + \mathbf{Q}$ has all its eigenvalues with positive real parts.
2. The following inequality holds:

$$t_{adv}\|\mathbf{Q}\|\frac{\|\mathbf{P}\| + \|\mathbf{Q}\|}{\|\mathbf{P} + \mathbf{Q}\|} < 1\,.$$

*Proof.* We recall the operator $\mathcal{T}$ as defined in Definition 3.1 and we will prove conditions under which $\mathcal{T}\mathbf{y}(t) \to \mathbf{0}_{m,1}$. We have:

$$\|\mathcal{T}\mathbf{y}(t)\| = \left\|\mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\mathbf{y}(0) + \int_0^t \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}](t-h)}\boldsymbol{g}(\mathbf{y}, h)dh\right\|,$$

and

$$\|\mathcal{T}\mathbf{y}(t)\| \leq \|\mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t}\|\|\mathbf{y}(0)\| + \left\|\int_0^t \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}](t-h)}\boldsymbol{g}(\mathbf{y}, h)dh\right\|.$$

Note that $\mathbf{e}^{-[\mathbf{P}+\mathbf{Q}]t} \to \mathbf{0}_{m,m}$ if $\mathbf{P}+\mathbf{Q} \succ 0$, or equivalently, if the matrix $\mathbf{P}+\mathbf{Q}$ has all its eigenvalues with positive real parts. This is the first assumption required and given by the theorem. We have for the second term in the above inequality:

$$\left\|\int_0^t \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}](t-h)}\boldsymbol{g}(\mathbf{y}, h)dh\right\| \leq$$

$$\int_0^t \mathbf{e}^{-\|\mathbf{P}+\mathbf{Q}\|(t-h)}\|\mathbf{Q}\|\Big(\int_h^{h+t_{adv}}[\|\mathbf{P}\|\|\mathbf{y}(u)\|+\|\mathbf{Q}\|\|\mathbf{y}(u+t_{adv})\|]du\Big)dh\,,$$

or, equivalently,

$$\Big\|\int_0^t \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}](t-h)}\boldsymbol{g}(\mathbf{y},h)dh\Big\|\leq$$

$$\int_0^\delta \mathbf{e}^{-\|\mathbf{P}+\mathbf{Q}\|(t-h)}\|\mathbf{Q}\|\Big(\int_h^{h+t_{adv}}[\|\mathbf{P}\|\|\mathbf{y}(u)\|+\|\mathbf{Q}\|\|\mathbf{y}(u+t_{adv})\|]du\Big)dh+$$
$$\int_\delta^t \mathbf{e}^{-\|\mathbf{P}+\mathbf{Q}\|(t-h)}\|\mathbf{Q}\|\Big(\int_h^{h+t_{adv}}[\|\mathbf{P}\|\|\mathbf{y}(u)\|+\|\mathbf{Q}\|\|\mathbf{y}(u+t_{adv})\|]du\Big)dh\,.$$

Under the assumption $\mathbf{P}+\mathbf{Q}>0$, we have:

$$\int_0^\delta \mathbf{e}^{-\|\mathbf{P}+\mathbf{Q}\|(t-h)}\|\mathbf{Q}\|\Big(\int_h^{h+t_{adv}}[\|\mathbf{P}\|\|\mathbf{y}(u)\|+\|\mathbf{Q}\|\|\mathbf{y}(u+t_{adv})\|]du\Big)dh<\frac{\epsilon}{2}\,.$$

In addition, by using the Definition 2.1:

$$\int_\delta^t \mathbf{e}^{-\|\mathbf{P}+\mathbf{Q}\|(t-h)}\|\mathbf{Q}\|\Big(\int_h^{h+t_{adv}}[\|\mathbf{P}\|\|\mathbf{y}(u)\|+\|\mathbf{Q}\|\|\mathbf{y}(u+t_{adv})\|]du\Big)dh\leq$$
$$\frac{\epsilon}{2}t_{adv}\|\mathbf{Q}\|\frac{\|\mathbf{P}\|+\|\mathbf{Q}\|}{\|\mathbf{P}+\mathbf{Q}\|}\,,$$

whereby using the second assumption given by the theorem, we get:

$$\int_\delta^t \mathbf{e}^{-\|\mathbf{P}+\mathbf{Q}\|(t-h)}\|\mathbf{Q}\|\Big(\int_h^{h+t_{adv}}[\|\mathbf{P}\|\|\mathbf{y}(u)\|+\|\mathbf{Q}\|\|\mathbf{y}(u+t_{adv})\|]du\Big)dh\leq\frac{\epsilon}{2}\,.$$

Hence we have that $\|\mathcal{T}\mathbf{y}(t)\|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$, and consequently $\mathcal{T}\mathbf{y}(t)\to\mathbf{0}_{m,1}$. Next we will prove that there exists $q\in[0,1)$ such that:

$$\|\mathcal{T}\mathbf{y}-\mathcal{T}\mathbf{w}\|\leq q\|\mathbf{y}-\mathbf{w}\|\,,$$

for all $\mathbf{y}$ and $\mathbf{w}$ bounded continuous columns of functions on $[0,+\infty)$. We have:

$$\|\mathcal{T}\mathbf{y}-\mathcal{T}\mathbf{w}\|=\Big\|\int_0^t \mathbf{e}^{-[\mathbf{P}+\mathbf{Q}](t-h)}\big(\boldsymbol{g}(\mathbf{y},h)-\boldsymbol{g}(\mathbf{w},h)\big)dh\Big\|\leq$$
$$\Big\|\int_0^t \mathbf{e}^{-\|[\mathbf{P}+\mathbf{Q}]\|(t-h)}\|\big(\boldsymbol{g}\big(\mathbf{y},h\big)-\boldsymbol{g}(\mathbf{w},h)\big)\|dh\Big\|.$$

Where

$$\|\big(\boldsymbol{g}(\mathbf{y},h)-\boldsymbol{g}(\mathbf{w},h)\big)\|=$$

$$\left\|\mathbf{Q}\int_h^{h+t_{adv}}[\mathbf{P}(\mathbf{y}(u)-\mathbf{w}(u))+\mathbf{Q}(\mathbf{y}(u+t_{adv})-\mathbf{w}(u+t_{adv}))]du\right\|\leq$$

$$\|\mathbf{Q}\|\int_h^{h+t_{adv}}[\|\mathbf{P}\|\|\mathbf{y}(u)-\mathbf{w}(u)\|+\|\mathbf{Q}\|\|\mathbf{y}(u+t_{adv})-\mathbf{w}(u+t_{adv})\|]du\leq$$

$$\|\mathbf{Q}\|\int_h^{h+t_{adv}}[\|\mathbf{P}\|+\|\mathbf{Q}\|]du\|\mathbf{y}-\mathbf{w}\|=t_{adv}\|\mathbf{Q}\|[\|\mathbf{P}\|+\|\mathbf{Q}\|]\|\mathbf{y}-\mathbf{w}\|\,,$$

or, equivalently,

$$\|\big(\boldsymbol{g}(\mathbf{y},h)-\boldsymbol{g}(\mathbf{w},h)\big)\|\leq t_{adv}\|\mathbf{Q}\|[\|\mathbf{P}\|+\|\mathbf{Q}\|]\|\mathbf{y}-\mathbf{w}\|\,.$$

Hence:

$$\|\mathcal{T}\mathbf{y}-\mathcal{T}\mathbf{w}\|\leq t_{adv}\|\mathbf{Q}\|\frac{\|\mathbf{P}\|+\|\mathbf{Q}\|}{\|\mathbf{P}+\mathbf{Q}\|}\|\mathbf{y}-\mathbf{w}\|\,,$$

whereby setting $q=t_{adv}\|\mathbf{Q}\|\frac{\|\mathbf{P}\|+\|\mathbf{Q}\|}{\|\mathbf{P}+\mathbf{Q}\|}$ and using the second assumption of the theorem, we arrive at the desired result. Hence, $\mathcal{T}$ is a contraction operator on $\mathbf{y}$ since there exists $q\in[0,1)$. In addition we proved that under the two assumptions of the theorem we have $\mathcal{T}\mathbf{y}(t)\to\mathbf{0}_{m,1}$. Thus, from contraction mapping principle we have that $\mathbf{y}(t)\to\mathbf{0}_{m,1}$. The proof is completed.

## Discussion on $s$-Domain Analysis

We apply the Laplace transform $\mathcal{L}$ into (3) and we get:

$$s\mathbf{Y}(s)-\mathbf{y}(0)+\mathbf{P}\mathbf{Y}(s)+\mathbf{Q}\int_0^{+\infty}\mathbf{y}(t+t_{adv})e^{-st}dt=\mathbf{0}_{m,1}\,,$$

where $\mathbf{Y}(s)$, $s\in\mathbb{C}$ is the Laplace transform of $\mathbf{y}(t)$. Equivalently:

$$s\mathbf{Y}(s)+\mathbf{P}\mathbf{Y}(s)+\mathbf{Q}\int_0^{+\infty}\mathbf{y}(t+t_{adv})e^{-st}dt=\mathbf{y}(0)\,.$$

By setting $t+t_{adv}=u$, we get:

$$(s\boldsymbol{I}_m+\mathbf{P})\mathbf{Y}(s)+\mathbf{Q}\int_{t_{adv}}^{+\infty}\mathbf{y}(u)e^{-s(u-t_{adv})}du=\mathbf{y}(0)\,,$$

or, equivalently,

$$(s\boldsymbol{I}_m+\mathbf{P})\mathbf{Y}(s)+\mathbf{Q}\int_0^{+\infty}\mathbf{y}(t)e^{-s(t-t_{adv})}dt-\mathbf{Q}\int_0^{t_{adv}}\mathbf{y}(t)e^{-s(t-t_{adv})}dt=\mathbf{y}(0)\,,$$

or, equivalently,

$$(s\boldsymbol{I}_m + \mathbf{P})\mathbf{Y}(s) + e^{st_{adv}}\mathbf{Q}\int_0^{+\infty} \mathbf{y}(t)e^{-st}dt - e^{st_{adv}}\mathbf{Q}\int_0^{t_{adv}} \mathbf{y}(t)e^{-st}dt = \mathbf{y}(0)\,,$$

or, equivalently,

$$(s\boldsymbol{I}_m + \mathbf{P} + e^{st_{adv}}\mathbf{Q})\mathbf{Y}(s) = \mathbf{y}(0) + e^{st_{adv}}\mathbf{Q}\int_0^{t_{adv}} \mathbf{y}(t)e^{-st}dt\,. \tag{12}$$

Note that one may be tempted to study the system via a system of neutral delayed differential equations, i.e. by setting $t + t_{adv} = u$ in (3). However, this is rather an alternative way to arrive to (12), as we show below:

$$\mathbf{y}'(u - t_{adv}) + \mathbf{P}\mathbf{y}(u - t_{adv}) + \mathbf{Q}\mathbf{y}(u) = \mathbf{0}_{m,1}\,, \quad u \geq t_{adv}\,,$$

i.e. a system of neutral delay differential equations. If we apply the Laplace transform $\mathcal{L}$ into this expression, we get:

$$\int_{t_{adv}}^{+\infty} \mathbf{y}'(u-t_{adv})e^{-su}du + \mathbf{P}\int_{t_{adv}}^{+\infty} \mathbf{y}(u-t_{adv})e^{-su}du + \mathbf{Q}\int_{t_{adv}}^{+\infty} \mathbf{y}(u)e^{-su}du = \mathbf{0}_{m,1}$$

whereby setting $u - t_{adv} = t$ we get:

$$\int_0^{+\infty} \mathbf{y}'(t)e^{-st}dt + \mathbf{P}\int_0^{+\infty} \mathbf{y}(t)e^{-st}dt + \mathbf{Q}\int_0^{+\infty} \mathbf{y}(t + t_{adv})e^{-st}dt = \mathbf{0}_{m,1}\,,$$

which with similar steps as previously leads to (12).

Studying the stability per se of system (3) through (12) requires first to deal with the quantity $\int_0^{t_{adv}} \mathbf{y}(t)e^{-st}dt$. With this regard, one may study relevant special cases, which implies that additional initial conditions are imposed to the original system. Depending on the special case examined, the time-advance $t_{adv}$ has a different impact on the system. As a matter of fact, for the special case $\mathbf{y}(t) = \mathbf{y}(t + kt_{adv})$, $k = 0, 1, 2, ...$, $\forall t \in [0, t_{adv}]$, $t_{adv}$ does not have any effect on the stability per se of the system, as we show in Proposition 3.1 and Remark 3.1.

**Proposition 3.1.** We consider system (3). Then, if $\mathbf{y}(t) = \mathbf{y}(t + kt_{adv})$, $k = 0, 1, 2, ...$, $\forall t \in [0, t_{adv}]$:

$$\mathbf{Y}(s) = \frac{1}{1 - e^{-st_{adv}}}\int_0^{t_{adv}} \mathbf{y}(t)e^{-st}dt.$$

*Proof.* We have that

$$\mathbf{Y}(s) = \int_0^{+\infty} \mathbf{y}(t)e^{-st}dt = \int_0^{t_{adv}} \mathbf{y}(t)e^{-st}dt + \int_{t_{adv}}^{2t_{adv}} \mathbf{y}(t)e^{-st}dt + \int_{2t_{adv}}^{3t_{adv}} \mathbf{y}(t)e^{-st}dt + ...$$

or, equivalently,

$$\mathbf{Y}(s) = \sum_{k=0}^{+\infty} \int_{kt_{adv}}^{(k+1)t_{adv}} \mathbf{y}(t)e^{-st}dt,$$

whereby setting $t = u + kt_{adv}$ we have

$$\mathbf{Y}(s) = \sum_{k=0}^{+\infty} \int_0^{t_{adv}} \mathbf{y}(u + kt_{adv})e^{-s(u+kt_{adv})}du,$$

or, equivalently, by using the assumption of the proposition

$$\mathbf{Y}(s) = \sum_{k=0}^{+\infty} \int_0^{t_{adv}} \mathbf{y}(u)e^{-s(u+kt_{adv})}du,$$

or, equivalently,

$$\mathbf{Y}(s) = \sum_{k=0}^{+\infty} \int_0^{t_{adv}} \mathbf{y}(t)e^{-s(t+kt_{adv})}dt,$$

or, equivalently,

$$\mathbf{Y}(s) = \sum_{k=0}^{+\infty} e^{-skt_{adv}} \int_0^{t_{adv}} \mathbf{y}(t)e^{-st}dt,$$

or, equivalently,

$$\mathbf{Y}(s) = \int_0^{t_{adv}} \mathbf{y}(t)e^{-st}dt \sum_{k=0}^{+\infty} \left(\frac{1}{e^{st_{adv}}}\right)^k.$$

and from here by using $\sum_{k=0}^{+\infty} \left(\frac{1}{e^{st_{adv}}}\right)^k = \frac{1}{1-e^{-st_{adv}}}$ we arrive at the result. The proof is completed.

**Remark 3.1.** By using Proposition 3.1, we can rewrite (12) as follows:

$$(s\boldsymbol{I}_m + \mathbf{P} + e^{st_{adv}}\mathbf{Q})\mathbf{Y}(s) = \mathbf{y}(0) + e^{st_{adv}}\mathbf{Q}(1 - e^{-st_{adv}})\mathbf{Y}(s)\,,$$

or, equivalently,

$$(s\boldsymbol{I}_m + \mathbf{P} + e^{st_{adv}}\mathbf{Q})\mathbf{Y}(s) = \mathbf{y}(0) + (e^{st_{adv}} - 1)\mathbf{Q}\mathbf{Y}(s)\,.$$

or, equivalently,

$$(s\boldsymbol{I}_m + \mathbf{P} + \mathbf{Q})\mathbf{Y}(s) = \mathbf{y}(0)\,.$$

# 4 Numerical Examples

In this section we provide two numerical examples. These examples are employed to illustrate our theoretical results, in particular the application of Theorem 3.1, under the effect of which the stability of any steady state of a system in the form of (3) is guaranteed.

## 4.1 Example 1

Consider the following non-causal system:

$$\mathbf{y}'(t) + \mathbf{P}\mathbf{y}(t) + \mathbf{Q}\mathbf{y}(t + t_{adv}) = \mathbf{0}_{m,1}\,, \tag{13}$$

where the coefficient matrix of the variables affected by the time advance $t_{adv}$ is

$$\mathbf{Q} = \begin{bmatrix} 0 & 1.135 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.415 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.25 \end{bmatrix},$$

while the coefficient matrix of the variables affected by the present time is

$$\mathbf{P} = \begin{bmatrix} 1.62 & 0 & -0.38 & -0.885 & 0.885 \\ 0 & 2.125 & 0 & -0.375 & 0.375 \\ -1.52 & 5.165 & 0.48 & 0 & 2.415 \\ 0 & -0.375 & 0 & 3.125 & -0.125 \\ 0 & 0.75 & 0 & 1.75 & 0 \end{bmatrix}.$$

For stability, the first condition of Theorem 3.1 requires that all eigenvalues of the matrix $\mathbf{P}+\mathbf{Q}$ are positive. Indeed, $\mathbf{P}+\mathbf{Q}$ has 5 eigenvalues, $\lambda_1 = 0.1$,

$\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 2.5$, $\lambda_5 = 3$, all positive, and thus the first condition is satisfied. Moreover, the second condition of Theorem 3.1 requires that:

$$t_{adv}\|\mathbf{Q}\|\frac{\|\mathbf{P}\| + \|\mathbf{Q}\|}{\|\mathbf{P} + \mathbf{Q}\|} < 1\,.$$

In our case, we have that $||\mathbf{P}|| = 7.598$, $||\mathbf{Q}|| = 2.947$ and $||\mathbf{P} + \mathbf{Q}|| = 8.149$. Where $||\cdot||$ denotes the Frobenius norm. Therefore, we find that

$$\|\mathbf{Q}\|\frac{\|\mathbf{P}\| + \|\mathbf{Q}\|}{\|\mathbf{P} + \mathbf{Q}\|} = 2.739\,,$$

and hence, we find that the system is guaranteed to be stable for

$$t_{adv} < 0.365\ s\,.$$

### 4.2 Example 2

In this example we consider system (13) with the following coefficient matrices:

$$\mathbf{P} = \begin{bmatrix} 2 & -0.75 - 1.875j & 0 & 1.25j & 3 + 6.25j & 1.125 + 3.125j \\ 0 & 3 - 1.25j & 1.25j & -2.5j & -2 - 20.5j & -0.75 + 25.75j \\ 0 & -3 + 11.25j & 2 - 1.25j & -2 + 2.5j & 14 - 47.5j & 4 - 16.25j \\ 0 & -1.5 - 3.75j & -0.5 - 1.25j & 2 + 2.5j & 6.5 + 5.5j & 2.125 + 8j \\ 0 & 0 & 0 & 0 & 2.5 - 7j & 0 + 3.5j \\ 0 & 0 & 0 & 0 & 0 & 2.5 + 7j \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & -0.25 - 0.625j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $j$ is the imaginary unit. In this case, only the third variable in the first equation of the system is impacted by the time-advance. The first condition of Theorem 3.1 is satisfied, since the real parts of all eigenvalues of the matrix $\mathbf{P}+\mathbf{Q}$ are positive. In particular, the matrix has 6 eigenvalues, $\lambda_1 = 2$, $\lambda_2 = 3+5j$, $\lambda_3 = 3-5j$, $\lambda_4 = 1$, $\lambda_5 = 2.5+7j$, $\lambda_5 = 2.5-7j$. Then, from the second condition of Theorem 3.1 we have that the system is stable for all time-advance values $t_{adv}$ for which

$$t_{adv} < t_{adv}^{\mathrm{m}} = \frac{\|\mathbf{P} + \mathbf{Q}\|}{\|\mathbf{Q}\|(\|\mathbf{P}\| + \|\mathbf{Q}\|)} = 1.471\ s\,.$$

We carry out a parametric analysis to show how changing the coefficient of the time-advanced variable affects the maximum time-advance $t^{\mathrm{m}}_{adv}$ for which the system is guaranteed to be stable, according to Theorem 3.1. To this aim, we assume that the non-zero element of $\mathbf{Q}$ is replaced by a generic complex number $\alpha+\beta j$, where $\alpha, \beta \in \mathbb{R}$. Then:

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & \alpha+\beta j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 1 shows how the maximum time-advance $t^{\mathrm{m}}_{adv}$ changes as $\alpha+\beta j$ is varied. The projections of the obtained surface onto to the $a$-$t^{\mathrm{m}}_{adv}$, $b$-$t^{\mathrm{m}}_{adv}$, and $a$-$b$ planes, are also depicted in the same plot. Notice that, for smaller values of $\alpha$, $\beta$, the value of $t^{\mathrm{m}}_{adv}$ increases. This is as expected, with the limit case being $\alpha = \beta = 0$, for which the time-advance has no effect on the system, and thus the system is always stable, or equivalently, $t^{\mathrm{m}}_{adv} \to \infty$ (which can be also confirmed by substituting $||\mathbf{Q}|| = 0$ in the second condition of Theorem 3.1).

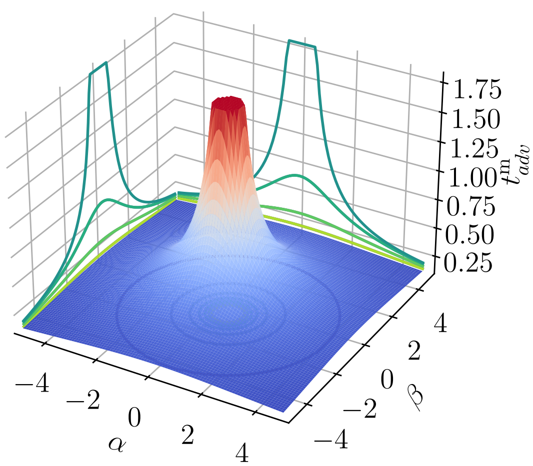


Figure 1: $\alpha$ versus $\beta$ versus $t^{\mathrm{m}}_{adv}$, Example 2.

### 4.3 Example 3

In this example we provide a simple application of our theory in electric power engineering. In particular, we consider a synchronous electric machine connected through a line to a bus of constant frequency and voltage (infinite bus). We also assume that the machine is equipped with an automatic controller aimed at damping electromechanical oscillations. In power engineering, such controller is known as power system stabilizer (PSS).

We assume that the machine's behavior is described by the classical model:

$$\begin{aligned} \delta'(t) &= \omega_b(\omega(t) - 1)\,, \\ M\omega'(t) &= P_{\mathrm{m}} - \frac{e_{\mathrm{q},1} v}{X} \sin(\delta(t) - \theta) - D(\omega(t) - 1)\,, \end{aligned} \tag{14}$$

where $\delta$, $\omega$ are the machines' rotor angle and speed; $P_{\mathrm{m}}$ is the machine's mechanical power output; $M$ is the machine's mechanical starting time and $D$ its damping coefficient; $\omega_b$ is the nominal synchronous angular frequency in rad/s; $v$, $\theta$ are the constant voltage magnitude and angle at the infinite bus; $e_{\mathrm{q},1}$ is the machine's internal electromotive force, which is also assumed constant; $X$ represents the sum of the machine's transient reactance and the line reactance.

Linearizing (14) around an equilibrium $[\delta_o \ \ \omega_o]^{\mathrm{T}}$ leads to:

$$\Delta\delta' = \omega_b \Delta\omega\,, \tag{15}$$

$$M\Delta\omega' = -\frac{e_{\mathrm{q},1} v \cos(\delta_o - \theta)}{X_{\mathrm{tot}}} \Delta\delta - D\Delta\omega\,, \tag{16}$$

In its simplest form, a PSS tracks the machine's speed and, following a certain control law, it introduces artificial damping into (16), which leads to:

$$M\Delta\omega' = -\frac{e_{\mathrm{q},1} v \cos(\delta_o - \theta)}{X_{\mathrm{tot}}} \Delta\delta - D\Delta\omega - u(\Delta\omega)\,. \tag{17}$$

For the purposes of this example, we assume that the PSS implements an *ideal* predictive control structure which is able to track the machine's rotor speed at a future time $t+t_{adv}$. The output of such predictive PSS is described by the following *time-advanced control law*:

$$u = K\Delta\omega(t + t_{adv})\,, \tag{18}$$

where $K$ is the control gain. Combining (15), (17), and (18) in a matrix form, we get:

$$\begin{bmatrix}\Delta\delta \\ \Delta\omega\end{bmatrix}'(t) = \begin{bmatrix} 0 & \omega_b \\ -\frac{e_{\mathrm{q},1} v\cos(\delta_o-\theta)}{MX_{\mathrm{tot}}} & -\frac{D}{M}\end{bmatrix}\begin{bmatrix}\Delta\delta \\ \Delta\omega\end{bmatrix}(t) - \begin{bmatrix}0 & 0 \\ 0 & \frac{K}{M}\end{bmatrix}\begin{bmatrix}\Delta\delta \\ \Delta\omega\end{bmatrix}(t+t_{adv}), \tag{19}$$

which equivalently, can be written as system in the form of (3), with:

$$\mathbf{y} = \begin{bmatrix}\Delta\delta \\ \Delta\omega\end{bmatrix}, \ \mathbf{P} = -\begin{bmatrix} 0 & \omega_b \\ -\frac{e_{\mathrm{q},1} v\cos(\delta_o-\theta)}{MX_{\mathrm{tot}}} & -\frac{D}{M}\end{bmatrix}, \ \mathbf{Q} = \begin{bmatrix}0 & 0 \\ 0 & \frac{K}{M}\end{bmatrix}$$

Let us give a numerical example. Let $e_{\mathrm{q},1} = 1.22$ pu[1], $v = 1$ pu, $\theta = 0$ rad, $P_{\mathrm{m}} = 1$ pu, $X = 0.7$ pu, $M = 5$ MWs/MVA, $\omega_b = 100\pi$ rad/s (50 Hz system).

The equilibrium of (14) is:

$$\begin{bmatrix}\delta_o \\ \omega_o\end{bmatrix} = \begin{bmatrix}\arcsin\left(\frac{P_{\mathrm{m}}X}{ve_{\mathrm{q}},1}\right) \\ 1\end{bmatrix} = \begin{bmatrix}0.61 \\ 1\end{bmatrix}$$

and matrix $\mathbf{P}$ becomes:

$$\mathbf{P} = \begin{bmatrix}0 & -314.16 \\ 0.29 & 0.4\end{bmatrix},$$

We check the conditions of Theorem 3.1 for control gain $K = 10$. In this case, matrix $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{bmatrix}0 & 0 \\ 0 & 2\end{bmatrix}.$$

The eigenvalues of $\mathbf{P}+\mathbf{Q}$ are $1.2 \pm 9.4j$, with the real part being positive. We also have that

$$\|\mathbf{Q}\|\frac{\|\mathbf{P}\|+\|\mathbf{Q}\|}{\|\mathbf{P}+\mathbf{Q}\|} = 2,$$

and hence, when $K = 10$, from Theorem 3.1 the system is guaranteed to be stable for

$$t_{adv} < 0.5\ s.$$

We finally consider that the PSS gain $K$ is an adjustable parameter and draw in Fig. 2 the variation of $t_{adv}^{\mathrm{m}}$ as $K$ changes. We note first that, for $K \geq 2$ the first condition of Theorem 3.1 is violated and second, that, as expected, when $K \to 0$, $t_{adv}^{\mathrm{m}} \to \infty$.

[1] per unit system (pu); in power engineering, quantities are often expressed as fractions of defined base units.

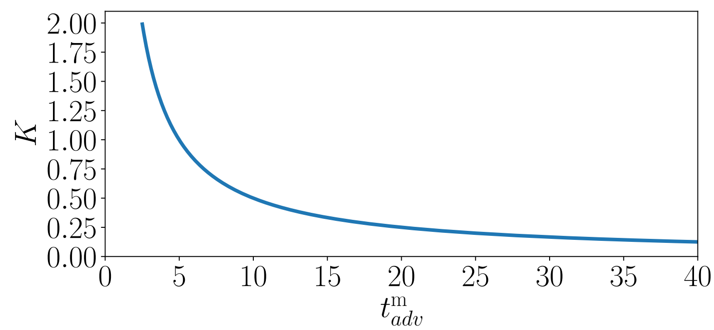


Figure 2: $t^{\mathrm{m}}_{adv}$ versus control gain $K$, Example 3.

## 5 Conclusions

In this article we defined a class of non-causal systems of differential equations and proved easily testable conditions under which any state of the system is stable. These results were justified with two numerical examples. As a future direction we aim to further extend these theoretical results and examine promising relevant applications, including the study and better understanding of predictive models and control systems. For all this there is already some research in progress.

## Data availability statement

The authors do not have any data related to this article.


## Acknowledgement

This work was supported by Sustainable Energy Authority of Ireland (SEAI), by funding Ioannis Dassios, and Federico Milano under Grant No. RDD/00681; and by the Swiss National Science Foundation, by funding Georgios Tzounas under NCCR Automation (grant no. 51NF40 18054).


## References


[1] O. Bazighifan, I. Dassios, On the Asymptotic Behavior of Advanced Differential Equations with a Non-Canonical Operator. Applied Sciences, **10**(9) 3130 (2020).

[2] D.M. Boore, S. Akkar, Effect of causal and acausal filters on elastic and inelastic response spectra Earthquake engineering & structural dynamics. **32**(11), 1729–1748 (2003)

[3] F.E. Cellier, E. Kofman, *Continuous System Simulation* (Springer, New York, 2006)

[4] N.R. Chaudhuri, S. Ray, R. Majumder, B. Chaudhuri, A new approach to continuous latency compensation with adaptive phasor power oscillation damping controller (POD), IEEE Trans. Power Systems, **25**, 939–946 (2010)

[5] L. Dai, *Notes in Control and information Sciences*, M. Thoma, A. Wyner (Eds.), Lecture, Singular Control Systems (Springer, Berlin, 1988)

[6] I. Dassios, O. Bazighifan, Oscillation Conditions for Certain Fourth-Order Non-Linear Neutral Differential Equation. Symmetry, **12**(7), 1096 (2020).

[7] Dassios I., Zimbidis A., Kontzalis C. The Delay Effect in a Stochastic Multiplier–Accelerator Model. Journal of Economic Structures, **3**(7), pp. 1-24 (2014).

[8] S. Devasia, Degang Chen, B. Paden. Nonlinear inversion-based output tracking. IEEE TAC, **41**(7), 930–942 (1996)

[9] A. De Cheveigné, I. Nelken, Filters: when, why, and how (not) to use them. Neuron. **102**(2), 280–293 (2019)

[10] J. Hale, *Theory of Functional Differential Equations* (Springer-Verlag, New York, 1977)

[11] M. Kitano, T. Nakanishi, K. Sugiyama, Negative group delay and superluminal propagation: An electronic circuit approach. IEEE Journal of selected Topics in Quantum electronics, **9**(1), 43–51 (2003)

[12] F. Milano, I. Dassios, Small-Signal Stability Analysis for Non-Index 1 Hessenberg Form Systems of Delay Differential-Algebraic Equations. IEEE Transactions on Circuits and Systems I: Regular Papers, **63**(9), 1521–1530 (2016).

[13] M. Liu, I. Dassios, G. Tzounas, F. Milano, Model-Independent Derivative Control Delay Compensation Methods for Power Systems.Energies, **13**(2), 342 (2020) DOI: 10.3390/en13020342

[14] M. Liu, I. Dassios, F. Milano, Delay margin comparisons for power systems with constant and time-varying delays. Electric Power Systems Research, **190**, 106627 (2021).

[15] M. Liu, I. Dassios, F. Milano, On the Stability Analysis of Systems of Neutral Delay Differential Equations. Circuits, Systems and Signal Processing, **38**(4), 1639-1653 (2019).

[16] M. Liu, I. Dassios, G. Tzounas, F. Milano, Stability Analysis of Power Systems with Inclusion of Realistic–Modeling of WAMS Delays. IEEE Transactions on Power Systems, **34**, (1), 627–636 (2019).

[17] T. Nakanishi, K. Sugiyama, M. Kitano, Demonstration of negative group delays in a simple electronic circuit, American Journal of Physics. American Association of Physics Teachers, **70**(11), 1117–1121 (2002)

[18] R. Rosen, *Anticipatory Systems: Philosophical, Mathematical, and Methodological Foundations* (Springer-Verlag, New York, 2012)

[19] S. Santra, A. Ghosh, I. Dassios, Second–order impulsive differential systems with mixed delays: Oscillation theorems. Mathematical Methods in the Applied Sciences. (2022).

[20] G. Tzounas, I. Dassios, F. Milano, Small-signal stability analysis of implicit integration methods for power systems with delays. Electric Power Systems Research, **211**, 108266 (2022).

[21] A. Widmann, E. Schröger, Filter effects and filter artefacts in the analysis of electrophysiological data Frontiers in psychology. Frontiers, **3**, 233 (2012)

[22] X. Yu, J. Jiang, Analysis and compensation of delays in fieldbus control loop using model predictive control, IEEE Trans. Instrum. Meas. **63**, 24322446 (2014)