Effect of Uniformly Distributed Parameter Line Models on the Evaluation of PV Curves and of the Maximum Loading Condition

R. Benato, *Senior Member, IEEE*, G. Gardan, *Member, IEEE,* F. Milano, *Fellow, IEEE*

***Abstract*— PV curves are generally obtained by considering lumped models of transmission lines. This approximated model can yield an inaccurate estimation of the maximum loading condition of the system. This letter shows that accuracy can be improved by considering line models with uniformly distributed parameters. Analytical evaluations of the PV curve and of the voltage collapse point of a two-bus system are obtained by applying the Ossanna's theorem. Then the impact of different line models on large-scale systems is evaluated through a continuation power flow analysis of a real-world model of the Sicilian transmission system including the Sicily-Malta 120 km cable connection.**

***Index Terms*—Ossanna's theorem, transmission line modelling, PV curves, voltage collapse, continuation power flow.**

I. Nomenclature

A. Variable and Parameters

R	Longitudinal lumped resistance
X	Longitudinal lumped reactance
$\underline{Z}$	Complex impedance
$\underline{S}$	Complex power
$\underline{U}$	Voltage phasor
$\underline{I}$	Current phasor
P	Active power
Q	Reactive power
d	Line length
φ	Load angle
$\underline{\boldsymbol{M}}$	Transmission matrix
ξ_{oss}	1^{st} Ossanna's parameter
u_y	2^{nd} Ossanna's parameter
χ	3^{rd} Ossanna's parameter

B. Subscripts

S	Sending-end terminal
R	Receiving-end terminal
0	No-load

II. Introduction

THE maximum loading condition and voltage stability assessment of a power system can be conveniently represented through bifurcation diagrams, which are often called PV or *nose* curves [1]-[4]. The study of the maximum loading condition flourished by the end of 1980s. However, the nonlinear nature of the network voltage collapse phenomenon and the increasing complexity of power systems, make this field of research an evergreen, e.g., [1]-[9]. Recent works, e.g. [10]-[12], all stress the importance for modern power systems with high shares of renewable and large sets of measurements of having accurate evaluations of the voltage collapse point. Our work precisely addresses this issue. The accurate determination of the PV curve and of the voltage collapse point, in fact, is crucial in power systems security analysis. In fact, even the dynamics of the voltage collapse physical phenomenon can be assessed accurately by considering the static behavior of the voltage as a function of the power [3]. The analytical expression of the PV curves is not possible in general. For radial two-bus systems, analytic expressions have been obtained by considering simplified electrical line models. For example, in [4], [6], [13] only the longitudinal inductive reactance X is taken into account, whereas in [1], [7] the impedance $\underline{Z} = R + jX$ is considered. Moreover, in the models available in the literature, the shunt admittances are generally neglected. The objectives of this letter are twofold. First, an analytic expression of the PV curve is determined for a two-bus system that considers a line model with uniformly distributed parameters. Then it is shown that detailed models of the transmission lines may have a significant impact on the estimation of maximum loading condition. In fact, the conventional lumped models utilized in voltage stability studies are unrealistic for long transmission lines (especially for cables) and distribution lines (high r/x ratios) [14]. The analytic formulation of the PV curves for the distributed parameter modelling is based on the Ossanna's theorem [5], [15]-[17]. The formulation is accurate and, we believe, also simple and elegant.

III. Classical PV Curve Formulations

In this section, we present two conventional analytic expressions of the PV curves for the two-bus system operating under balanced three-phase, steady-state sinusoidal conditions (see Fig. 1). These expressions serve for the comparison with the novel formulation based on the Ossanna's theorem, which is duly presented in Section IV.

A. Lossless Lumped Line Model

Due to the predominantly inductive behavior, electrical lines can be modelled as pure longitudinal reactive links, where the value of the lumped reactance is $X = x \cdot d$, with x the kilometric reactance and d the line length. Considering a two-bus system, the relation between the active power and the voltage at the receiving terminal can be formulated as in [6]:

R. Benato (e-mail: roberto.benato@unipd.it), G. Gardan (e-mail: giovanni.gardan@phd.unipd.it) are with the Department of Industrial Engineering, University of Padova, via Gradenigo 6/a 35131.

F. Milano (e-mail: federico.milano@ucd.ie) is with the School of Electrical and Electronic Engineering, University College Dublin, D04 V1W8 Dublin, Ireland.

$$p_R = \frac{|\underline{u}_R|^2}{x}\frac{-\tan(\varphi)+\sqrt{\tan^2(\varphi)-\left(1-|\underline{u}_S|^2/|\underline{u}_R|^2\right)}}{1+\tan^2(\varphi)} \qquad (1)$$

Hereinafter, the lossless lumped formulation is indicated as "X" method.

B. Lossy Lumped Line Model

If the line resistance is not neglected, the electrical line can be modelled as a lumped longitudinal impedance, $\underline{Z} = \underline{z}\ d = R + jX$, where $\underline{z} = r + jx$ is the p.u. longitudinal kilometric impedance of the line, and d is the line length. The lossy lumped line model is currently utilized to make model-based voltage stability evaluations on power systems [1]. By considering a two-bus system, the relation between the active power and voltage can be formulated as it follows [7]:

$$|\underline{u}_R|^4 + \left(2\,p_R r + 2\,q_R x - |\underline{u}_S|^2\right)|\underline{u}_R|^2 + |\underline{s}_R|^2\,|\underline{z}|^2 = 0 \qquad (2)$$

Hereinafter, the lossy lumped formulation is indicated as "RX" method. We note that the inclusion of the shunt elements (π-model), for the formulation of the PV curve is generally not considered in the literature. Its formulation is not reported here since it is conceptually similar to (2).

IV. Two-Bus System Collapse Point and PV Curve Analytical Determination by the Ossanna's Theorem

The Ossanna's theorem states that the balanced, fundamental frequency steady-state operating conditions of any line supplying a load absorbing a complex power $\underline{S}_R$ can be determined analytically. The theorem can be demonstrated by combining the pu formulation and the Thevenin theorem [5]. Hence, an accurate curve formulation addressing the uniformly distributed resistive, inductive, capacitive, and conductive behavior is considered [5].

With this aim, let us consider a transmission line with uniformly distributed parameters (see Fig. 1). The line can be modelled by its Thevenin equivalent circuit as seen from the receiving-end [5], where its no-load voltage generator and impedance depend on the transmission matrix coefficients as in the following:

$$\underline{Z} = \frac{\underline{B}}{\underline{A}} \quad and \quad \underline{U}_{0R} = \frac{\underline{U}_S}{\underline{A}} \qquad (3)$$

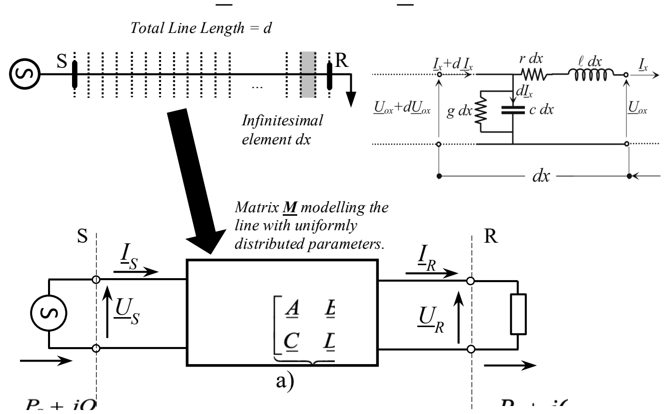

Fig. 1 Uniformly distributed parameter model of the line.

where $\underline{\boldsymbol{M}}$ is the transmission matrix, *i.e.*,

$$\underline{\boldsymbol{M}} = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix}. \qquad (4)$$

By means of the Ossanna's theorem, the upper $\underline{u}_1$ and the lower $\underline{u}_2$ PV curves have the following expressions:

$$\begin{aligned} \underline{u}_1 &= [(1/2) + \xi_{oss}] - j u_y \\ \underline{u}_2 &= [(1/2) - \xi_{oss}] - j u_y \end{aligned} \qquad (5)$$

where the expressions of ξ_{oss} and u_y are defined as follows [5]:

$$\xi_{oss} = \sqrt{\frac{1}{4} - s\cdot\cos(\chi) - (s\cdot\sin\chi)^2}$$

$$s = |\underline{s}|$$

$$\chi = \angle\underline{z} - \angle\underline{s}$$

$$u_y = s\cdot\sin\chi$$

Hereinafter, the uniformly distributed formulation in (5) is indicated as "BG" method. It is worth reminding that $\xi_{oss} \in \sim^{+}$ is the condition describing the line physical feasibility condition, *i.e.*,

$$\xi_{oss} \geq 0 \qquad (6)$$

In this letter we are also interested in the analytical determination of the voltage-collapse point. With this aim, by imposing $\xi_{oss} = 0$, that is the condition for which the expression of $\underline{u}_1$ and $\underline{u}_2$ are equal (see (5)), it yields:

$$\frac{1}{4} - s\cdot\cos(\chi) - (s\cdot\sin\chi)^2 = 0. \qquad (7)$$

Assuming a constant power factor (φ=const.) and the following writing for the complex power: $\underline{s}_R = p_R + j p_R\cdot\tan(\varphi)$, (7) can be written as a second-order function of p:

$$\left[(1+\tan^2(\varphi))\sin^2\chi\right]\cdot p_R^2 + \left[\sqrt{(1+\tan^2(\varphi))^2}\cos(\chi)\right]\cdot p_R - \frac{1}{4} = 0 \qquad (8)$$

which admits the following two solutions:

$$p_{R1,2} = \frac{-\sqrt{1+\tan^2(\varphi)}\,\cos(\chi) \pm \sqrt{\left(\sqrt{1+\tan^2(\varphi)}\,\cos(\chi)\right)^2 + (1+\tan^2(\varphi))\sin^2\chi}}{2(1+\tan^2(\varphi))\sin^2\chi} \qquad (9)$$

From the identity $\cos^2\chi + \sin^2\chi = 1$, considering only the positive solution of (9), and collecting the term $\sqrt{1+\tan^2(\varphi)}$ the maximum loading condition at the voltage collapse point is obtained as:

$$p_{max} = \frac{\sqrt{1+\tan^2(\varphi)}\,(1-\cos(\chi))}{2(1+\tan^2(\varphi))\sin^2\chi}. \qquad (10)$$

Then, from p_{max} the voltage collapse value can be computed by means of (5) ($\underline{u}_1$ or $\underline{u}_2$ can be used since ξ_{oss}=0). If the slack generator reactive power limit q^{max} is considered, when $q_S < q^{max}$, the system is always described by means of the upper expression $\underline{u}_1$ of (5). However, when it reaches the reactive power limit $q_S = q^{max}$, the slack generator model changes to a QV model, where Q = q^{max} [6]. Therefore, the PV curve that considers the generator reactive power limits can be analytically determined:

$$\underline{u}_R = 1 - \underline{z}\left(\left(p_R + \frac{q_{max} - p_R \tan(\varphi)}{\tan(\angle \underline{z})}\right) - j\, q_{max}\right). \qquad (11)$$

As it is well-known and shown in Section V, generator reactive power limits can reduce the loading margin of the system (critical limit-induced bifurcations).

V. Case Study

This section illustrates the effect of the line models on the determination of the PV curves of power systems. To illustrate the analytic expression discussed in the previous sections, first a two-bus connection is considered. Fig. 2 shows the PV curves (*pf*=0.9 lagging) for the long overhead line (OHL) (see Fig. 2a) and the Sicily-Malta interconnection cable (IC) (see Fig. 2b) respectively using the X, BG, and RX models. The differences among the BG, X, and RX methods are synthesized in Fig. 2, which shows the different shapes of the PV curves. If one assumes that the Ossanna's model is the reference, Fig. 2a shows that the X model underestimates the voltage collapse point by 11.74%, whereas the RX one underestimates the voltage collapse point by 3.52%. For the long IC, on the other hand, Fig. 2b shows that the X model overestimates the voltage collapse point by 16.5%, whereas the RX underestimates the voltage collapse point by 6.4%. For a unitary power factor, this error reaches 57%. Results in Fig. 2 also indicate that the BG method PV curves correctly explains the voltage rise at the end of the line when it transmits small active power (near to the no-load condition i.e., P = 0). This behavior cannot be observed with the X and RX methods. Finally, the results show that the different voltage collapse points of the three models depend on the line typology (i.e., OHL, IC or low/medium voltage lines), and its length.

TABLE I

Kilometric Parameters of Three Real Transmission Lines Belonging to the Sicilian Network (OHL: Overhead Line, IC: Insulated Cable)

Typology	r [Ω/km]	ℓ [mH/km]	g [nS/km]	c [nF/km]	d [km]
(b) Long OHL	0.021	0.86	0	13.41	218.5
(c) Long IC	0.077	0.46	52	164	120

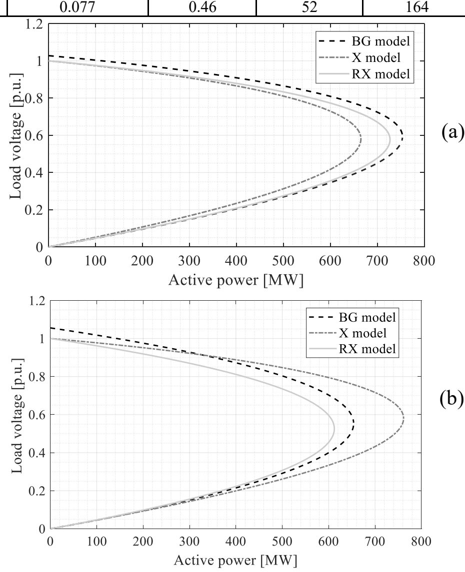

Fig. 2. PV curve (power factor=0.9) comparisons among BG, X and RX for a) a 218.5 km OHL Italian line, and b) the 120 km Sicily-Malta cable.

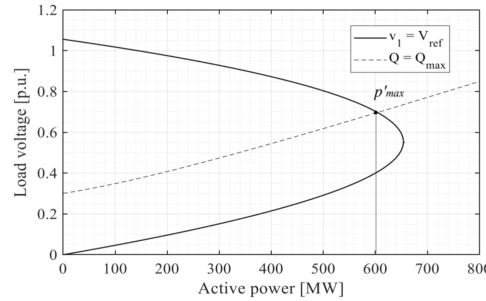

Fig. 3. PV curve (power factor=0.9) considering generator reactive power limit (Q_{max} = 1000 Mvar) for the 120 km Sicily-Malta cable.

For example, a parametric analysis confirms that by varying the line lengths, for the IC the X model always overestimates the voltage collapse point, due to the cable high capacitive nature, which is properly and fully considered only by the BG model.

Fig. 3 shows the PV curve (BG method) by considering the reactive power limit (fixed to 1000 Mvar) for the generator supplying the Sicily-Malta cable. The consideration of the reactive power limit brings to a reduction of the maximum transmittable active power p'_{max}.

For completeness, Fig. 4 shows the results of the continuation power flow (CPF) analysis for the real-world model of the Sicily-Malta system. This model includes 102 buses, 75 lines, 59 transformers and 14 generators. The CPF analysis has been carried using the Python-based software tool Dome [18].

The continuation power flow analysis is a homotopy technique that parametrizes the power flow problem with a scalar loading level [19]. The loading level scales both active and reactive powers of all loads as well as the active power of all generators. A distributed slack variable is utilized to share loss increments among all generators. Note that, in Fig. 4, the loading level normalized with respect to the initial loading condition, i.e., the loading level is 1 at the initial operating point. Finally, the PV curves shown in Fig. 4 are obtained considering the reactive power limits of the generators as well as the three models discussed in this letter, namely RX, X and BG. The maximum loading conditions correspond to a saddle-node bifurcation and are 2.643, 2.688 and 2.625 times the base-case operating point for the RX, X and BG models, respectively. The BG model is the one that leads to the lowest loading margin, thus indicating that the conventional lumped model of the transmission lines might not lead to conservative results.

VI. Conclusions

The letter discusses the effect of line modelling on the maximum loading conditions of power systems. Results show that considering accurate models of the lines does not always have the same impact, i.e., it is not always conservative. The letter also proposes an analytical formulation of PV curves for two bus-systems based on the Ossanna's theorem and the steady-state model of transmission lines with uniformly distributed parameter one. A numerical appraisal between the Ossanna's curve (BG curve taken as the reference) and the ones found in the technical literature (X and RX curves) are performed, underlining that different line models can lead to significantly different maximum loading conditions. Finally, the impact of the BG line model is compared with the X and

RX models for a real-world model of the meshed Sicilian network, which confirms that importance of an accurate modelling of transmission lines, especially for long cables such as the Sicily-Malta connection.

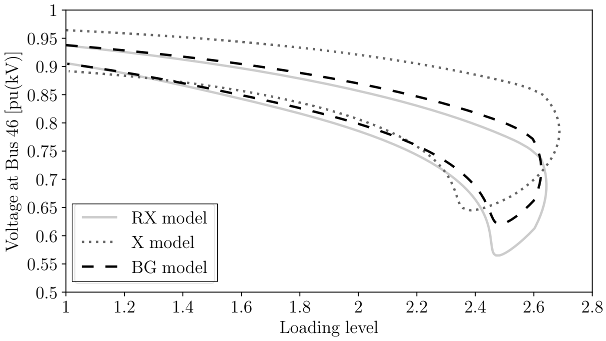

Fig. 4. PV curve obtained with the detailed model of the Italy-Malta power system. Bus 46 is the receiving-end bus of the Sicily-Malta connection.

VII. References

[1] M. G. O'Grady and M. A. Pai, "Analysis of voltage collapse in power systems," in *Proc. 21st Annu. North American Power Symp.*, Rolla, MO, USA, Oct. 9-10, 1989, pp. 151–160.

[2] V. Ajjarapu and C. Christy, "The continuation power flow: a tool for steady state voltage stability analysis," in *Proc. Conf. Papers 1991 Power Industry Computer Application Conf.*, Baltimore, MD, USA, May 7-9, 1991, pp. 304–311

[3] I. Dobson, H. Glavitsch, C.-C. Liu, Y. Tamura, and K. Vu, "Voltage collapse in power systems," *IEEE Circuits Devices Mag.*, vol. 8, no. 3, pp. 40–45, May 1992.

[4] M. K. Pal, "Voltage stability conditions considering load characteristics," *IEEE Trans. on Power Systems*, vol. 7, no. 1, pp. 243-249, Feb. 1992.

[5] R. Benato and S. D. Sessa, "The Ossanna's Theorem for Educational Purposes: Impact of Distributed Parameter Transmission Lines on Power Systems," *IEEE Access*, vol. 9, pp. 10577–10585, Jan. 2021.

[6] F. Milano, "Continuation Power Flow Analysis," in *Power system modelling and scripting*, Springer-Verlag London Limited, 2010.

[7] J. Modarresi, E. Gholipour, and A. Khodabakhshian, "A comprehensive review of the voltage stability indices", *Renew. Sustain. Energy Rev.*, vol. 63, pp. 1–12, Sept. 2016.

[8] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Voltage collapse in complex power grids," *Nat. Commun.*, vol. 7, no. 1, pp. 1–8, Feb. 2016.

[9] T. v. Cutsem and C. Vournas, *Voltage Stability of Electric Power Systems*. Springer Science+Business Media, 2008.

[10] A. O. Muhammed, M. Rawa, "A systematic PVQV-curves approach for Investigating the impact of solar-photolvoltaic generation in power system using PowerWorld simulator,", *Energies*, vol. 13, no. 13, May 2020.

[11] R. Pourbagher, S.Y. Derakhshanded, and M. E. H. Golsham, "A novel method for online voltage stability assessment based on PMU measurements and Thevenin equivalent," *IET Gener., Transmiss. & Distribution,* vol. 16, no. 9, pp. 1780-1794, Dec. 2021.

[12] N. Hossinzadeh, A. Aziz, A. Mahmud, A. Gargoom, M. Rabbani, "Voltage stability of power systems with renewable-energy inverter-based generators: a review," *Electronics*, vol. 10, no. 2, pp. 1-27, Jan. 2021.

[13] F. Saccomanno, *Electric Power Systems: Analysis and Control.* John Wiley & Sons Inc, 2003.

[14] M. S. M. Koksal, "Remark on the lumped parameter modeling of transmission lines," *Electr. Mach. Power Syst.*, vol. 28, no. 6, pp. 565–575, June 2000.

[15] J. Ossanna, *Neue Arbeitsdiagramme über die Spannungsänderung in Wechselstromnetzen*. Engineering societies library, 1926.

[16] J. Ossanna, "Fernübertragungsmöglichkeiten großer Energiemengen," *ETZ*, vol. 43, p. 1025, 1922.

[17] A. P. R. Benato and A. Paolucci, *EHV AC undergrounding electrical power*, Springer-Verlag London Limited, 2010.

[18] F. Milano, "A Python-based software tool for power system analysis," in *2013 IEEE Power & Energy Society General Meeting*, Vancouver, BC, Canada, Jul. 21-25, 2013, pp. 1-5.

[19] IEEE/PES Power System Stability Subcommittee, "Voltage Stability Assessment: Concepts, Practices and Tools," Technical Report PES-TR9, August 2003.