Singular over–determined systems of linear differential equations

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Abstract: We study a class of singular over–determined systems of differential equations and firstly prove that there exist solutions under certain conditions. For the case of existence we provide closed formulas of solutions and based on the spectrum of the pencil of the system we study uniqueness of solutions. Then, we extend these results to higher order singular overdetermined systems. Finally, we use this type of systems to model electrical power systems and provide numerical examples.

Keywords : Singular systems, non-square matrices, existence and uniqueness of solutions, power systems.

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1 Introduction

We consider the following singular system of differential equations:

$$\mathbf{E}\,\dot{\boldsymbol{x}}(t) = \mathbf{A}\,\boldsymbol{x}(t) + \boldsymbol{\omega}(t)\,. \tag{1}$$

The elements of the matrix coefficients in this system are assumed constant, and $\mathbf{E}, \mathbf{A} \in \mathbb{C}^{r\times m}$, $r > m$. In addition $\boldsymbol{x} : [0, +\infty] \mapsto \mathbb{C}^{m\times 1}$, and $\boldsymbol{\omega} : [0, +\infty] \mapsto \mathbb{C}^{r\times 1}$.

If we apply the Laplace $\mathcal{L}$ transform, we get:

$$\mathbf{E}\mathcal{L}\Big\{\dot{\boldsymbol{x}}(t)\Big\} = \mathbf{A}\mathcal{L}\{\boldsymbol{x}(t)\} + \mathcal{L}\{\boldsymbol{\omega}(t)\}\,,$$

or, equivalently,

$$\mathbf{E}(s\boldsymbol{X}(s) - \boldsymbol{x}_o) = \mathbf{A}\boldsymbol{X}(s) + \boldsymbol{\Omega}(s)\,,$$

or, equivalently,

$$(s\mathbf{E} - \mathbf{A})\boldsymbol{X}(s) = \mathbf{E}\boldsymbol{x}_o + \boldsymbol{\Omega}(s)\,.$$

With $\boldsymbol{X}(s)$, $\boldsymbol{\Omega}(s)$ we denote the functions after $\mathcal{L}$ is applied into $\boldsymbol{x}(t)$, $\boldsymbol{\omega}(t)$ respectively, and $\boldsymbol{x}_o = \boldsymbol{x}(0)$. From the above equation, it is obvious that the polynomial matrix $s\mathbf{E} - \mathbf{A}$ is essential in studying (1). This polynomial is defined as the *pencil* of (1), see [16].

Singular systems of differential equations, see [2, 6, 8, 13, 15, 17], and difference equations, see [9, 18] have attracted the interest of several researchers in the last few decades. Some interesting results have also been obtained for singular systems of equations evolving fractional operators, see [1, 7, 12, 14, 24]. This type of systems appear in control theory, see [5, 11, 23], and in several applications in electrical engineering such as the modeling of electrical circuits, see [17], and power system dynamics, see [19, 20, 22].

Despite several studies, most articles deal with singular systems that have regular pencils. The regularity of the pencil means that the matrices are square $r = m$, while the pencil formed has a determinant not identically equal to zero. Focus is then given in studying the solutions and stability of such a system through the eigenvalues of this pencil.

Singular systems with singular pencils are usually avoided. There are two types of singular pencils. A first case is the matrix coefficients of the system to be square but with a pencil that has a determinant identically zero. Meaning that the pencil is not invertible, something that is crucial for the existence of solutions of the system that appears in the frequency domain after the Laplace transform is applied to the system in the time domain. The other type of singular pencil is the matrix coefficients to be non-square. In this case the determinant of the pencil can not be defined.

In this article we study singular over–determined systems of linear differential equations. The pencil of this type of systems is singular. Unlike the regular pencil which may have finite eigenvalues & an infinite eigenvalue, a singular pencil has additional invariants the minimal column and row minimal indices. This type of invariants for such a pencil are not always easy to be obtained. It becomes even more complicated when dealing with large scale systems. Another important characteristic of this case considered is that existence of solutions for a system with a singular pencil is not automatically satisfied. This is very important for many applications for which the model is significant only for certain range of its parameters. In these cases a careful interpretation of results or even a redesign of the system maybe needed. In this paper we propose a method that analyses this type of singular pencils, and derives explicit and easily testable conditions for which the system has a solution.

After using this result based on our proposed method in which we can identify existence of solutions for these systems, we use the spectrum of the pencil to prove a result for uniqueness of solutions.

To sum up, this article is organized as follows. In Section 2, we will firstly study the existence, and uniqueness of solutions of system (1). In the same section, we will also extend the results to higher order systems. Then in Section 3, we will construct examples of electrical power system modelling and we will close the article by providing numerical examples in Section 4.

2 Existence, Uniqueness of solutions, and Formulas

As seen in the previous section, after applying $\mathcal{L}$ into (1), by assuming that $\boldsymbol{x_o}$ is unknown, and setting $\boldsymbol{x_o} = \boldsymbol{C} \in \mathbb{C}^{m\times 1}$, we arrive at:

$$(s\mathbf{E} - \mathbf{A})\boldsymbol{X}(s) = \mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)\,. \tag{2}$$

When the algebraic system is over–determined, $r > m$, see [10], a matrix function $\hat{\boldsymbol{P}}(s)$, $\hat{\boldsymbol{P}} : \mathbb{C} \mapsto \mathbb{R}^{r\times r}$ (which can be computed via the Gauss-Jordan Elimination Method, see [21]), can be defined in such a way that if it is multiplied to the pencil $s\mathbf{E} - \mathbf{A}$ it provides the following results:

$$\hat{\boldsymbol{P}}(s)(s\mathbf{E} - \mathbf{A}) = \begin{bmatrix} \hat{\boldsymbol{A}}(s) \\ \mathbf{0}_{r_1,m} \end{bmatrix}, \quad \text{with} \quad \hat{\boldsymbol{P}}(s) = \begin{bmatrix} \hat{\boldsymbol{P}}_1(s) \\ \hat{\boldsymbol{P}}_2(s) \end{bmatrix}, \tag{3}$$

where $\hat{\boldsymbol{A}} : \mathbb{C} \mapsto \mathbb{R}^{m_1\times m}$, with $m_1 + r_1 = r$, is a matrix such that if $[\hat{a}_{ij}(s)]_{1\leq i\leq m_1}^{1\leq j\leq m}$ are its elements, for $i = j$ all elements are non-zero and for $i \neq j$ all elements are zero and $\hat{\boldsymbol{P}}_1(s) \in \mathbb{R}^{m_1\times r}$, $\hat{\boldsymbol{P}}_2(s) \in \mathbb{R}^{r_1\times r}$. Hence by multiplying $\hat{\boldsymbol{P}}(s)$ to (2) we get

$$\hat{\boldsymbol{P}}(s)(s\mathbf{E} - \mathbf{A})\boldsymbol{X}(s) = \hat{\boldsymbol{P}}(s)[\mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)]\,,$$

or, equivalently, by using (3):

$$\begin{bmatrix} \hat{\boldsymbol{A}}(s) \\ \mathbf{0}_{r_1,m} \end{bmatrix} \boldsymbol{X}(s) = \begin{bmatrix} \hat{\boldsymbol{P}}_1(s) \\ \hat{\boldsymbol{P}}_2(s) \end{bmatrix} [\mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)]\,,$$

from where we get two subsystems:

1. $\hat{\boldsymbol{A}}(s)\boldsymbol{X}(s) = \hat{\boldsymbol{P}}_1(s)[\mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)]$;
2. $\mathbf{0}_{r_1,m}\boldsymbol{X}(s) = \hat{\boldsymbol{P}}_2(s)[\mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)]$,

with solution:

$$\boldsymbol{X}(s) = [\hat{\boldsymbol{A}}(s)]^{-1}\hat{\boldsymbol{P}}_1(s)[\mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)]\,,$$

if both

$$m_1 = m \quad \text{and} \quad \hat{\boldsymbol{P}}_2(s)[\mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)] = \mathbf{0}_{r_1,m} \tag{4}$$

hold. Hence, under the condition (4), the over–determined system (1) has solutions defined as:

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\{[\hat{\boldsymbol{A}}(s)]^{-1}\hat{\boldsymbol{P}}_1(s)[\mathbf{E}\boldsymbol{C} + \boldsymbol{\Omega}(s)]\}\,.$$

We will refer to (1) with $r > m$ as singular *over–determined* linear systems of differential equations.

Staying in this case, i.e. $r > m$ with (4) to hold, as mentioned above, the solution of (1) is given by

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\{[\hat{\boldsymbol{A}}(s)]^{-1}\hat{\boldsymbol{P}}_1(s)\mathbf{E}\}\boldsymbol{C} + \mathcal{L}^{-1}\{[\hat{\boldsymbol{A}}(s)]^{-1}\hat{\boldsymbol{P}}_1(s)\boldsymbol{\Omega}(s)\}\,.$$

All elements of the matrix $[\hat{\boldsymbol{A}}(s)]^{-1}\hat{\boldsymbol{P}}_1(s)\mathbf{E}$ are fractions of polynomials of the form $\frac{\Pi(s)}{\Theta(s)}$ with $deg\{\Pi(s)\} < deg\{\Theta(s)\}$, with *deg* being order of a polynomial. Hence it is easy to conclude that the inverse of $\mathcal{L}$ in the first term always exists. Hence, if we set $\boldsymbol{\Psi}_0(t) = \mathcal{L}^{-1}\{[\hat{\boldsymbol{A}}(s)]^{-1}\hat{\boldsymbol{P}}_1(s)\mathbf{E}\}$, and $\boldsymbol{\Psi}_1(t) = \mathcal{L}^{-1}\{[\hat{\boldsymbol{A}}(s)]^{-1}\hat{\boldsymbol{P}}_1(s)\}$, we get:

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}_0(t)\boldsymbol{C} + \int_0^\infty \boldsymbol{\Psi}_1(t-\tau)\boldsymbol{\omega}(\tau)d\tau\,. \tag{5}$$

To conclude we can state the following Theorem:

Theorem 1 (Existence of the solution)**.** *We assume system* (1) *with* $r > m$*. Then there exist solutions for* (1) *given by* (5) *if and only if* (3) *holds.*

In Theorem 1 we didn't refer to uniqueness of solutions. This is not always guaranteed for this type of systems, and could be studied easier by using the spectrum of this pencil. For the case that $r > m$ the invariants of $s\mathbf{E} - \mathbf{A}$ can be finite eigenvalues, an infinite eigenvalue, and row minimal indices, see [10, 16]. Let $\mathcal{N}_l(s\mathbf{E} - \mathbf{A})$ be the set of rational vector spaces with t=dim$\mathcal{N}_l(s\mathbf{E} - \mathbf{A})$, and as such they are spanned by minimal polynomial bases of minimal degrees (set of row minimal indices):

$$\zeta_1 = \zeta_2 = \cdots = \zeta_h = 0 < \zeta_{h+1} \leq \cdots \leq \zeta_{h+k=\beta}\,.$$

$\beta - h = k$ is the number of the indices that we are interested for. From the Kronecker theory, see [10, 16], the regular matrices $\mathbf{P}$, $\mathbf{P} \in \mathbb{C}^{r\times r}$, and

$\mathbf{Q}$, $\mathbf{Q} \in \mathbb{C}^{m\times m}$, can be defined in such a way that if tha have the following impact to the pencil $s\mathbf{E} - \mathbf{A}$, and the matrices $\mathbf{E}$, $\mathbf{A}$:

$$\begin{aligned} \mathbf{PEQ} = \mathbf{E}_K = \mathbf{I}_p \oplus \mathbf{H}_q \oplus \mathbf{E}_\zeta\,, \\ \mathbf{PAQ} = \mathbf{A}_K = \mathbf{J}_p \oplus \mathbf{I}_q \oplus \mathbf{A}_\zeta\,, \end{aligned} \tag{6}$$

where $\mathbf{J}_p$ is the Jordan matrix for the finite eigenvalues, $\mathbf{H}_q$ a nilpotent matrix with index q_* which is actually the Jordan matrix of the zero eigenvalue of the pencil $s\mathbf{A} - \mathbf{E}$. The matrices $\mathbf{E}_\zeta$, $\mathbf{A}_\zeta$ are defined as

$$\mathbf{E}_\zeta = \begin{bmatrix} \mathbf{I}_{\zeta_{h+1}} \\ \mathbf{0}_{1,\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{I}_{\zeta_{h+2}} \\ \mathbf{0}_{1,\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathbf{I}_{\zeta_{h+k}} \\ \mathbf{0}_{1,\zeta_{h+k}} \end{bmatrix}\,,$$

$$\mathbf{A}_\zeta = \begin{bmatrix} \mathbf{0}_{1,\zeta_{h+1}} \\ I_{\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0}_{1,\zeta_{h+2}} \\ \mathbf{I}_{\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathbf{0}_{1,\zeta_{h+k}} \\ I_{\zeta_{h+k}} \end{bmatrix},$$

with $p + q + \sum_{i=1}^{k}[\zeta_{h+i}] + k = r$, $p + q + \sum_{i=1}^{k}[\zeta_{h+i}] = m$. In addition, let:

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_p & \mathbf{Q}_q & \mathbf{Q}_\zeta \end{bmatrix},$$

with $\mathbf{P}_1 \in \mathbb{C}^{p\times r}$, $\mathbf{P}_2 \in \mathbb{C}^{q\times r}$, $\mathbf{P}_3 \in \mathbb{C}^{\zeta_1\times r}$, $\zeta_1 = k + \sum_{i=1}^{k}[\zeta_{h+i}]$ and $\mathbf{Q}_p \in \mathbb{C}^{m\times p}$, $\mathbf{Q}_q \in \mathbb{C}^{m\times q}$, $\mathbf{Q}_\zeta \in \mathbb{C}^{m\times \zeta_2}$ and $\zeta_2 = \sum_{i=1}^{k}[\zeta_{h+i}]$. Let $\boldsymbol{x}(t) = \mathbf{Q}\boldsymbol{z}(t)$, then:

$$\mathbf{EQ}\,\dot{\boldsymbol{z}}(t) = \mathbf{AQ}\,\boldsymbol{z}(t) + \boldsymbol{\omega}(t)\,,$$

or, equivalently,

$$\mathbf{PEQ}\,\dot{\boldsymbol{z}}(t) = \mathbf{PAQ}\,\boldsymbol{z}(t) + \mathbf{P}\boldsymbol{\omega}(t)\,,$$

or, equivalently,

$$\begin{aligned} \dot{\boldsymbol{z}}_p(t) = \mathbf{J}_p\boldsymbol{z}_p(t) + \mathbf{P}_1\boldsymbol{\omega}(t)\,, \\ \mathbf{H}_q\,\dot{\boldsymbol{z}}_q(t) = \boldsymbol{z}_q(t) + \mathbf{P}_2\boldsymbol{\omega}(t)\,, \end{aligned}$$

and

$$\mathbf{E}_\zeta\dot{\boldsymbol{z}}_\zeta(t) = \mathbf{A}_\zeta\boldsymbol{z}_\zeta(t) + \mathbf{P}_3\boldsymbol{\omega}(t)\,.$$

Where $\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{z}_p(t) \\ \boldsymbol{z}_q(t) \\ \boldsymbol{z}_\zeta(t) \end{bmatrix}$, $\boldsymbol{z}_p(t) \in \mathbb{C}^{p\times 1}$, $\boldsymbol{z}_p(t) \in \mathbb{C}^{q\times 1}$ and $\boldsymbol{z}_\zeta(t) \in \mathbb{C}^{\zeta_2\times 1}$. These systems have the following solutions:

$$\boldsymbol{z}_p(t) = \mathrm{e}^{\mathbf{J}_p t}\boldsymbol{C} + \int_0^\infty \mathrm{e}^{\mathbf{J}_p(t-\tau)}\boldsymbol{\omega}(\tau)d\tau\,,$$

and

$$\boldsymbol{z}_q(t) = -\sum_{i=0}^{q_*-1} \mathbf{H}_q^i \mathbf{P}_2 \frac{d^i}{dt^i} \boldsymbol{\omega}(t) \,.$$

For the third subsystem let

$$\boldsymbol{z}_\zeta(t) = \begin{bmatrix} \boldsymbol{z}_{\zeta_{h+1}}(t) \\ \boldsymbol{z}_{\zeta_{h+2}}(t) \\ \vdots \\ \boldsymbol{z}_{\zeta_{h+k}}(t) \end{bmatrix}, \quad \boldsymbol{z}_{\zeta_{h+i}}(t) \in \mathbb{C}^{(\zeta_{h+i})\times 1}, \quad i = 1, 2, \dots, k \,, \tag{7}$$

with

$$\boldsymbol{z}_{\zeta_{h+i}}(t) = \begin{bmatrix} z_{\zeta_{h+i},1}(t) \\ z_{\zeta_{h+i},2}(t) \\ \vdots \\ z_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix},$$

and

$$\mathbf{P}_3 \boldsymbol{\omega}(t) = \begin{bmatrix} \boldsymbol{\Omega}_1(t) \\ \boldsymbol{\Omega}_2(t) \\ \vdots \\ \boldsymbol{\Omega}_k(t) \end{bmatrix}, \quad \boldsymbol{\Omega}_i(t) \in \mathbb{C}^{(\zeta_{h+i}+1)\times 1}, \quad i = 1, 2, \dots, k \,,$$

with

$$\boldsymbol{\Omega}_i(t) = \begin{bmatrix} u_{i0} \\ u_{i1} \\ u_{i2} \\ \vdots \\ u_{i\zeta_{h+i}} \end{bmatrix}, \quad i = 1, 2, \dots, k \,.$$

By replacing and using these relations we get

$$\begin{bmatrix} \mathbf{I}_{\zeta_{h+i}} \\ \mathbf{0}_{1,\zeta_{h+i}} \end{bmatrix} \dot{\boldsymbol{z}}_{\zeta_{h+i}}(t) = \begin{bmatrix} \mathbf{0}_{1,\zeta_{h+i}} \\ \mathbf{I}_{\zeta_{h+i}} \end{bmatrix} \boldsymbol{z}_{\zeta_{h+i}}(t) + \boldsymbol{\Omega}_i(t) \,,$$

or, equivalently, by using the above expressions

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_{\zeta_{h+i},1}(t) \\ \dot{z}_{\zeta_{h+i},2}(t) \\ \vdots \\ \dot{z}_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} z_{\zeta_{h+i},1}(t) \\ z_{\zeta_{h+i},2}(t) \\ \vdots \\ z_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix} + \begin{bmatrix} u_{i0} \\ u_{i1} \\ u_{i2} \\ \vdots \\ u_{i\zeta_{h+i}} \end{bmatrix},$$

or, equivalently,

$$\begin{array}{r} \dot{z}_{\zeta_{h+i},1}(t) = u_{i0}\,, \\ \dot{z}_{\zeta_{h+i},2}(t) = z_{\zeta_{h+i},1}(t) + u_{i1}\,, \\ \vdots \\ \dot{z}_{\zeta_{h+i},\zeta_{h+i}}(t) = z_{\zeta_{h+i},\zeta_{h+i}-1}(t) + u_{i(\zeta_{h+i}-1)}\,, \\ 0 = z_{\zeta_{h+i},\zeta_{h+i}}(t) + u_{i\zeta_{h+i}}\,. \end{array} .$$

We have a system of $\zeta_{h+i}+1$ differential equations and ζ_{h+i} unknown functions. If we denote the n-th order derivative with $\frac{d^n}{dt^n}$, for $n \geq 4$, we get:

$$\begin{array}{r} z_{\zeta_{h+i},\zeta_{h+i}}(t) = -u_{i\zeta_{h+i}}\,, \\ z_{\zeta_{h+i},\zeta_{h+i}-1}(t) = -u_{i(\zeta_{h+i}-1)} - \dot{u}_{i\zeta_{h+i}}\,, \\ z_{\zeta_{h+i},\zeta_{h+i}-2}(t) = -u_{i(\zeta_{h+i}-2)} - \dot{u}_{i(\zeta_{h+i}-1)} - \dfrac{d^2}{dt^2}u_{i\zeta_{h+i}}\,, \\ \vdots \\ z_{\zeta_{h+i},1}(t) = -u_{i1} - \dot{u}_{i2} - \ldots - \dfrac{d^{\zeta_{h+i}-1}}{dt^{\zeta_{h+i}-1}}u_{i\zeta_{h+i}}\,. \end{array} .$$

In order to solve the system we used the last ζ_j equations. By applying these results in the first equation we get

$$u_{i0} = -\dot{u}_{i1} - \frac{d^2}{dt^2}u_{i2} - \ldots - \frac{d^{\zeta_{h+i}}}{dt^{\zeta_{h+i}}}u_{i\zeta_{h+i}}\,,$$

or, equivalently,

$$\sum_{\rho=0}^{\zeta_{h+i}} \frac{d^\rho}{dt^\rho}u_{i\rho} = 0\,,$$

which is the condition for the over–determined system (1) to have solutions given by

$$\boldsymbol{x}(t) = \mathbf{Q}\boldsymbol{z}(t) = \begin{bmatrix} \mathbf{Q}_p & \mathbf{Q}_q & \mathbf{Q}_\zeta \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathbf{J}_p t}\boldsymbol{C} + \int_0^\infty \mathrm{e}^{\mathbf{J}_p(t-\tau)}\boldsymbol{\omega}(\tau)d\tau \\ -\sum_{i=0}^{q_*-1} \mathbf{H}_q^i \mathbf{P}_2 \frac{d^i}{dt^i}\boldsymbol{\omega}(t) \\ \boldsymbol{z}_\zeta \end{bmatrix},$$

or, equivalently,

$$\boldsymbol{x}(t) = \mathbf{Q}_p \left[\mathrm{e}^{\mathbf{J}_p t}\boldsymbol{C} + \int_0^\infty \mathrm{e}^{\mathbf{J}_p(t-\tau)}\boldsymbol{\omega}(\tau)d\tau \right] - \mathbf{Q}_q \sum_{i=0}^{q_*-1} \mathbf{H}_q^i \mathbf{P}_2 \frac{d^i}{dt^i}\boldsymbol{\omega}(t) + \mathbf{Q}_\zeta \boldsymbol{z}_\zeta \,.$$

We state the theorem:

Theorem 2 (Existence of the solution)**.** *There exist solutions for the overdetermined system* (1) *if and only if* (4) *holds. Then the general solution is given by*

$$\boldsymbol{x}(t) = \mathbf{Q}_p \left[\mathrm{e}^{\mathbf{J}_p t}\boldsymbol{C} + \int_0^\infty \mathrm{e}^{\mathbf{J}_p(t-\tau)}\boldsymbol{\omega}(\tau)d\tau \right] - \mathbf{Q}_q \sum_{i=0}^{q_*-1} \mathbf{H}_q^i \mathbf{P}_2 \frac{d^i}{dt^i}\boldsymbol{\omega}(t) + \mathbf{Q}_\zeta \boldsymbol{z}_\zeta \,, \tag{8}$$

where $\mathbf{Q}_p$*,* $\mathbf{J}_p$*,* $\mathbf{Q}_q$*,* $\mathbf{H}_q$*,* $\mathbf{P}_2$*,* $\mathbf{Q}_\zeta$ *are defined in* (6)*, and* $\boldsymbol{z}_\zeta$ *is given by* (7)*.*

Theorem 2 provides an alternative closed formula of solutions for (1) under the assumption that there exist solutions. This Theorem will also help us identify under which condition the solution of (1) can be unique. Initial conditions of the system which lead to a unique solution will be referred as consistent, while initial conditions of the system which lead to infinite solutions will be referred as non-consistent.

Corollary 1 (Uniqueness of the solution)**.** *Assume that system* (1) *has solutions given by* (8)*. Then if* $\boldsymbol{x}(t_o) = \boldsymbol{x}_o$ *is given, the solution is unique if and only if:*

$$\boldsymbol{x}_o \in \mathrm{colspan}\mathbf{Q}_p - \mathbf{Q}_q \sum_{i=0}^{q_*-1} \mathbf{H}_q^i \mathbf{P}_2 \frac{d^i}{dt^i}\boldsymbol{\omega}(0) + \mathbf{Q}_\zeta \boldsymbol{z}_\zeta \,.$$

In this case the initial conditions will be consistent. Otherwise, the initial conditions will be non-consistent and system (1) *will have infinite many solutions. Finally, for given consistent initial conditions, the constant column*

$\boldsymbol{C}$ *(that appears in the general solution of* (1)*) is the unique solution of the linear system*

$$\mathbf{Q}_p\boldsymbol{C} = \left[\boldsymbol{x}_o + \mathbf{Q}_q\sum_{i=0}^{q_*-1}\mathbf{H}_q^i\mathbf{P}_2\frac{d^i}{dt^i}\boldsymbol{\omega}(0) + \mathbf{Q}_\zeta\boldsymbol{z}_\zeta\right].$$

Proof. For $t = 0$ in (8) of theorem 2 we have:

$$\boldsymbol{x}(0) = \mathbf{Q}_p\boldsymbol{C} - \mathbf{Q}_q\sum_{i=0}^{q_*-1}\mathbf{H}_q^i\mathbf{P}_2\frac{d^i}{dt^i}\boldsymbol{\omega}(0) + \mathbf{Q}_\zeta\boldsymbol{z}_\zeta\,,$$

and from here we arrive easily at the desired condition. The proof is completed. $\square$

Higher order System

In this subsection we consider the following system of differential equations of higher order:

$$\mathbf{A}_n\,\frac{d^n}{dt^n}\boldsymbol{x}(t) + \mathbf{A}_{n-1}\,\frac{d^{n-1}}{dt^{n-1}}\boldsymbol{x}(t) + \cdots + \mathbf{A}_1\,\dot{\boldsymbol{x}}(t) + \mathbf{A}_0\,\boldsymbol{x}(t) = \boldsymbol{\Omega}(t)\;, \qquad (9)$$

where $\mathbf{A}_i \in \mathbb{C}^{r\times m}$, $r > m$, $i = 0, 1, \ldots, n$, are non-square matrices, and $\hat{\boldsymbol{x}} \in \mathbb{C}^{m\times 1}$, $\boldsymbol{\Omega} \in \mathbb{C}^{r\times 1}$. In this case the matrix pencil $s^n\mathbf{A}_n + s^{n-1}\mathbf{A}_{n-1} + \cdots + s\mathbf{A}_1 + \mathbf{A}_0$ is called singular since $r > m$. In the next lemma we will use the following notation. Let $[I_{ij}]_{i=1,2,\ldots,r}^{j=1,2,\ldots,m}$ be an element of the matrix $\mathbf{I}_{r,m}$ in the i-th row, j-th column, with $I_{ij} = 1$ for $i = j$, and $I_{ij} = 0$ for $i \neq j$.

Lemma 3 (Formulation of differential equations)**.** *System* (9) *can be reformulated in the following generalized system of differential equations of first order:*

$$\mathbf{E}\,\dot{\boldsymbol{x}}(t) = \mathbf{A}\,\boldsymbol{x}(t) + \boldsymbol{\omega}(t)\,, \qquad (10)$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{I}_{r,m} & \mathbf{0}_{r,m} & \ldots & \mathbf{0}_{r,m} & \mathbf{0}_{r,m} \\ \mathbf{0}_{r,m} & \mathbf{I}_{r,m} & \ldots & \mathbf{0}_{r,m} & \mathbf{0}_{r,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{r,m} & \mathbf{0}_{r,m} & \ldots & \mathbf{I}_{r,m} & \mathbf{0}_{r,m} \\ \mathbf{0}_{r,m} & \mathbf{0}_{r,m} & \ldots & \mathbf{0}_{r,m} & \mathbf{A}_n \end{bmatrix} \in \mathbb{C}^{nr\times nm}\,,$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{r,m} & \mathbf{I}_{r,m} & \dots & \mathbf{0}_{r,m} & \mathbf{0}_{r,m} \\ \mathbf{0}_{r,m} & \mathbf{0}_{r,m} & \dots & \mathbf{0}_{r,m} & \mathbf{0}_{r,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{r,m} & \mathbf{0}_{r,m} & \dots & \mathbf{0}_{r,m} & \mathbf{I}_{r,m} \\ -\mathbf{A}_0 & -\mathbf{A}_1 & \dots & -\mathbf{A}_{n-2} & -\mathbf{A}_{n-1} \end{bmatrix} \in \mathbb{C}^{nr\times nm}.$$

Furthermore where

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_{n-1} \\ \boldsymbol{x}_n \end{bmatrix} \in \mathbb{C}^{nm\times 1}, \quad \boldsymbol{\omega}(t) = \begin{bmatrix} \mathbf{0}_{r,1} \\ \mathbf{0}_{r,1} \\ \vdots \\ \mathbf{0}_{r,1} \\ \boldsymbol{\Omega}(t) \end{bmatrix} \in \mathbb{C}^{nr\times 1}, \quad \textit{and} \quad \boldsymbol{x}_1(t) = \hat{\boldsymbol{x}}(t)\,.$$

Proof. Firstly we set:

$$\begin{aligned} \boldsymbol{x}_1 &= \boldsymbol{x}\,, \\ \boldsymbol{x}_2 &= \dot{\boldsymbol{x}}\,, \\ &\vdots \\ \boldsymbol{x}_{n-1} &= \frac{d^{n-2}}{dt^{n-2}}\boldsymbol{x}\,, \\ \boldsymbol{x}_n &= \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{x}\,, \end{aligned}$$

whereby taking the derivatives we get

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \dot{\boldsymbol{x}}\,, \\ \dot{\boldsymbol{x}}_2 &= \frac{d^2}{dt^2}\boldsymbol{x}\,, \\ &\vdots \qquad\qquad . \\ \dot{\boldsymbol{x}}_{n-1} &= \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{x}\,, \\ \mathbf{A}_n\,\dot{\boldsymbol{x}}_n &= \mathbf{A}_n\,\frac{d^n}{dt^n}\boldsymbol{x}\,, \end{aligned}$$

or, equivalently,

$$\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \boldsymbol{x}_2(t)\,,\\
\dot{\boldsymbol{x}}_2 &= \boldsymbol{x}_3(t)\,,\\
&\vdots \\
\dot{\boldsymbol{x}}_{n-1} &= \boldsymbol{x}_n(t)\,,\\
\mathbf{A}_n\dot{\boldsymbol{x}}_n &= -\mathbf{A}_{n-1}\,\boldsymbol{x}_n - \cdots - \mathbf{A}_0\,\boldsymbol{x}_1 + \boldsymbol{\Omega}\,.
\end{aligned}$$

The above equations can then be written in the matrix form (10). The proof is completed. □

Theorem 4 (Equivalence between polynomial and linear pencils)**.** *The pencils* $s^n\mathbf{A}_n + s^{n-1}\mathbf{A}_{n-1} + \cdots + s\mathbf{A}_1 + \mathbf{A}_0$*,* $s\mathbf{E} - \mathbf{A}$ *of systems* (9)*,* (10) *respectively, have exactly the same finite eigenvalues.*

Proof. We will prove this theorem for a regular pencil. For a singular pencil the proof will be similar since the finite eigenvalues are obtained from similar sub-determinants to the regular pencil. Note that if $\mathbf{M}$ is a square matrix, and $\mathbf{M}_i$, $i = 1, 2, 3, 4$ are matrices (not necessary square) such that $\mathbf{M}$ can be written in the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4 \end{bmatrix},$$

then, see [16], if $\mathbf{M}_1$ is square and invertible

$$\det(\mathbf{M}) = \det(\mathbf{M}_1)\det(\mathbf{M}_4 - \mathbf{M}_3\mathbf{M}_1^{-1}\mathbf{M}_2)\,.$$

For $r = m$ the pencil $s\mathbf{E} - \mathbf{A}$ is equal to

$$s\mathbf{E} - \mathbf{A} = \begin{bmatrix} s\mathbf{I}_m & -\mathbf{I}_m & \ldots & \mathbf{0}_{m,m} & \mathbf{0}_{m,m} \\ \mathbf{0}_{m,m} & s\mathbf{I}_m & \ldots & \mathbf{0}_{m,m} & \mathbf{0}_{m,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{m,m} & \mathbf{0}_{m,m} & \ldots & s\mathbf{I}_m & -\mathbf{I}_m \\ \mathbf{A}_0 & \mathbf{A}_1 & \ldots & \mathbf{A}_{n-2} & s\mathbf{A}_n + \mathbf{A}_{n-1} \end{bmatrix}.$$

Hence if we set $\mathbf{M}(s) := s\mathbf{E} - \mathbf{A}$, the pencil can be written in the form:

$$s\mathbf{E} - \mathbf{A} = \begin{bmatrix} \mathbf{M}_1(s) & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4(s) \end{bmatrix},$$

where

$$\mathbf{M}_1(s) = \begin{bmatrix} s\mathbf{I}_m & -\mathbf{I}_m & \dots & \mathbf{0}_{m,m} \\ \mathbf{0}_{m,m} & s\mathbf{I}_m & \dots & \mathbf{0}_{m,m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{m,m} & \mathbf{0}_{m,m} & \dots & s\mathbf{I}_m \end{bmatrix} \in \mathbb{R}^{(n-1)m\times(n-1)m},$$

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{0}_{m,m} \\ \mathbf{0}_{m,m} \\ \vdots \\ -\mathbf{I}_m \end{bmatrix} \in \mathbb{R}^{(n-1)m\times m},$$

and

$$\mathbf{M}_3 = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \dots & \mathbf{A}_{n-2} \end{bmatrix} \in \mathbb{C}^{m\times(n-1)m}, \quad \mathbf{M}_4(s) = s\mathbf{A}_n + \mathbf{A}_{n-1} \in \mathbb{C}^{m\times m}.$$

Then since

$$\mathbf{M}_1^{-1}(s) = \begin{bmatrix} s^{-1}\mathbf{I}_m & s^{-2}\mathbf{I}_m & \dots & s^{-(n-1)}\mathbf{I}_m \\ \mathbf{0}_{m,m} & s^{-1}\mathbf{I}_m & \dots & s^{-(n-2)}\mathbf{I}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{m,m} & \mathbf{0}_{m,m} & \dots & s^{-1}\mathbf{I}_m \end{bmatrix} \in \mathbb{R}^{(n-1)m\times(n-1)m},$$

and

$$\mathbf{M}_1^{-1}(s)\mathbf{M}_2 = \begin{bmatrix} -s^{-(n-1)}\mathbf{I}_m \\ -s^{-(n-2)}\mathbf{I}_m \\ \vdots \\ -s^{-1}\mathbf{I}_m \end{bmatrix} \in \mathbb{R}^{(n-1)m\times m},$$

we have

$$\mathbf{M}_3\mathbf{M}_1^{-1}(s)\mathbf{M}_2 = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \dots & \mathbf{A}_{n-2} \end{bmatrix} \begin{bmatrix} -s^{-(n-1)}\mathbf{I}_m \\ -s^{-(n-2)}\mathbf{I}_m \\ \vdots \\ -s^{-1}\mathbf{I}_m \end{bmatrix},$$

or, equivalently,

$$\mathbf{M}_3\mathbf{M}_1^{-1}(s)\mathbf{M}_2 = -s^{-(n-1)}\mathbf{A}_0 - s^{-(n-2)}\mathbf{A}_1 - \dots - s^{-1}\mathbf{A}_{n-2}.$$

In addition:

$$\mathbf{M}_4(s) - \mathbf{M}_3\mathbf{M}_1^{-1}(s)\mathbf{M}_2 = s\mathbf{A}_n + \mathbf{A}_{n-1} + s^{-1}\mathbf{A}_{n-2} + \dots + s^{-(n-2)}\mathbf{A}_1 + s^{-(n-1)}\mathbf{A}_0.$$

Hence,

$$\det(s\mathbf{E}-\mathbf{A}) = \det(\mathbf{M}_1(s))\det(\mathbf{M}_4(s)-\mathbf{M}_3\mathbf{M}_1^{-1}(s)\mathbf{M}_2)\,,$$

or, equivalently,

$$\det(s\mathbf{E}-\mathbf{A}) = \det(s^{n-1}\mathbf{I}_m)\det(s\mathbf{A}_n+\mathbf{A}_{n-1}+s^{-1}\mathbf{A}_{n-2}+\ldots+s^{-(n-2)}\mathbf{A}_1+s^{-(n-1)}\mathbf{A}_0)\,,$$

or, equivalently,

$$\det(s\mathbf{E}-\mathbf{A}) = \det(s^n\mathbf{A}_n + s^{n-1}\mathbf{A}_{n-1} + \cdots + s\mathbf{A}_1 + \mathbf{A}_0)\,.$$

The proof is completed. □

Remark 5. *In lemma 3 we proved that system* (9) *can be reformulated into system* (10)*, and in theorem 4 we proved that the pencil of* (9) *has the same finite eigenvalues with the pencil of* (10)*. Consequently there exist solutions for system* (9) *if and only if there exist solutions for* (10)*. Hence there exist solutions for* (9) *if for system* (10)*, the condition* (4) *holds, see theorem 1.*

We consider now the matrices $\mathbf{P}$, $\mathbf{Q}$ as defined for the singular pencil $s\mathbf{E}-\mathbf{A}$ with $r > m$ in (6). Then we can define the matrices $\mathbf{Q}^1$, $\mathbf{Q}_p^1$, and $\mathbf{Q}_\zeta^1$ as

$$\begin{aligned} \mathbf{Q} &= \begin{bmatrix} \mathbf{Q}^1 \\ \mathbf{Q}^2 \end{bmatrix} \in \mathbb{C}^{nm\times nm}\,, \\ \mathbf{Q}_p &= \begin{bmatrix} \mathbf{Q}_p^1 \\ \mathbf{Q}_p^2 \end{bmatrix} \in \mathbb{C}^{nm\times p}\,, \\ \mathbf{Q}_\zeta &= \begin{bmatrix} \mathbf{Q}_\zeta^1 \\ \mathbf{Q}_\zeta^2 \end{bmatrix} \in \mathbb{C}^{nm\times \zeta_2}\,, \end{aligned} \tag{11}$$

where $\mathbf{Q}^1 \in \mathbb{C}^{m\times nm}$, $\mathbf{Q}_p^1 \in \mathbb{C}^{m\times p}$, and $\mathbf{Q}_\zeta^1 \in \mathbb{C}^{m\times \zeta_2}$.

Theorem 6 (Existence of the solution)**.** *Consider system* (9)*. Then there always exists a solution for* (9) *if for the pencil* $s\mathbf{E}-\mathbf{A}$ *of system* (10) *the condition* (4) *holds. In this case the solution is given by*

$$\boldsymbol{x}(t) = \mathbf{Q}_p^1\,\mathrm{e}^{\mathbf{J}_p(t)}\boldsymbol{C} + \mathbf{Q}^1\boldsymbol{K}(t) + \mathbf{Q}_\zeta^1\boldsymbol{z}_\zeta\,, \tag{12}$$

where $\boldsymbol{K}(t) = \begin{bmatrix} \int_0^t \mathrm{e}^{\mathbf{J}_p(t-s)}\mathbf{P}_1\,\boldsymbol{U}(s)ds \\ -\sum_{i=0}^{q_*-1}\mathbf{H}_q^i\mathbf{P}_2\,\frac{d^i}{dt^i}U(t) \end{bmatrix}$, $\boldsymbol{C} \in \mathbb{C}^{p\times 1}$ *is constant vector, and* $\mathbf{J}_p \in \mathbb{C}^{p\times p}$, $\mathbf{H}_q \in \mathbb{C}^{q\times q}$ *are the Jordan matrices related to the finite, infinite eigenvalues respectively. The matrices* $\mathbf{P}$, $\mathbf{Q}$ *are defined for the singular pencil* $s\mathbf{E}-\mathbf{A}$ *with* $r > m$ *in* (6)*. The matrices* $\mathbf{Q}^1$, $\mathbf{Q}_p^1$*, and* $\mathbf{Q}_\zeta^1$ *are defined in* (11)*.*

Proof. From Theorem 1, there always exist a solution for the over-determined system (10) if (4) holds. In this case its solution is given by (8):

$$\boldsymbol{x}(t) = \mathbf{Q}_p\, \mathrm{e}^{\mathbf{J}_p(t)}\boldsymbol{C} + \mathbf{Q}\boldsymbol{K}(t) + \mathbf{Q}_\zeta \boldsymbol{z}_\zeta\,.$$

By using lemma 3, if for system (9) $r > m$, and (4) holds for the pencil $s\mathbf{E} - \mathbf{A}$ of (10), then the solution of (9) is given by:

$$\boldsymbol{x}(t) = \mathbf{Q}_p^1\, \mathrm{e}^{\mathbf{J}_p(t)}\boldsymbol{C} + \mathbf{Q}^1\boldsymbol{K}(t) + \mathbf{Q}_\zeta^1 \boldsymbol{z}_\zeta\,.$$

The proof is completed. $\square$

Remark 7. *Let*

$$\boldsymbol{x}(0),\ \dot{\boldsymbol{x}}(0), \dots,\ \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{x}(0) \tag{13}$$

be the initial conditions of (9)*. We set*

$$\boldsymbol{x}(0) = \begin{bmatrix} \boldsymbol{x}(0) \\ \dot{\boldsymbol{x}}(0) \\ \vdots \\ \frac{d^{n-2}}{dt^{n-2}}\boldsymbol{x}(0) \\ \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{x}(0) \end{bmatrix} \in \mathbb{C}^{nm\times 1}.$$

If $r > m$, and (4) *holds, by using corollary 1 the solution of* (9) *is unique if and only if:*

$$\boldsymbol{x}(0) \in \mathrm{colspan}\mathbf{Q}_p + \mathbf{Q}\boldsymbol{K}(0) + \mathbf{Q}_\zeta \boldsymbol{z}_\zeta\,.$$

Then in the general solution (12) *of* (9)*, $\boldsymbol{C}$ is the unique solution of the linear system*

$$\mathbf{Q}_p\boldsymbol{C} = [\boldsymbol{x}(0) - \mathbf{Q}\boldsymbol{K}(0) - \mathbf{Q}_\zeta \boldsymbol{z}_\zeta]\,.$$

Remark 8. *Let $\mathcal{H}(t)$ be the Heaviside function and*

$$\kappa_1(t) = \mathcal{H}(t) - \mathcal{H}(-t) = \left\{ \begin{array}{ccc} 1 & , & t > 0 \\ 0 & , & t = 0 \end{array} \right\},$$

$$\kappa_2(t) = \mathcal{H}(-t) = \left\{ \begin{array}{ccc} 1 & , & t = 0 \\ 0 & , & t \neq 0 \end{array} \right\}.$$

Then, if the initial conditions (13) *are non-consistent, i.e. the conditions for consistency in remark 7 do not hold, then system* (10) *can be written as*

$$\kappa_1(t)\,(\mathbf{E}\,\dot{\boldsymbol{x}}(t) - \boldsymbol{\omega}(t)) = \mathbf{A}\,\boldsymbol{x}(t) - \kappa_2(t)\,\mathbf{A}\,\boldsymbol{x}(0)\,, \quad t \geq 0\,.$$

This is a generalized linear matrix differential equation of first order, and although the initial conditions are given due to their inconsistency the solution of this system is not unique. If $r > m$, and (4) *holds its solution is given by:*

$$\boldsymbol{x}(t) = \kappa_1(t)\,[\mathbf{Q}_p\,\mathrm{e}^{\mathbf{J}_p(t)}\boldsymbol{C} + \mathbf{Q}\boldsymbol{K}(t) + \mathbf{Q}_\zeta \boldsymbol{z}_\zeta] + \kappa_2(t)\,\boldsymbol{x}(0)\,, \quad t \geq 0\,.$$

In both cases where $\boldsymbol{C} = \begin{bmatrix} c_1 & c_2 & \ldots & c_p \end{bmatrix}^{\mathrm{T}}$ is constant vector, it can not be defined, and hence the dimension of the solution vector space is p. In addition, we can rewrite system (9) *in the following form:*

$$\sum_{i=0}^{n} \mathbf{A}_i\,\frac{d^i}{dt^i}\boldsymbol{x}(t) = \boldsymbol{\Omega}(t), \quad t > 0\,.$$

For $t \geq 0$ system (9) *can take the following form:*

$$\mathbf{A}_0\,\boldsymbol{x}(t) + \kappa_1(t)\,\left[\sum_{i=1}^{n} \mathbf{A}_i \frac{d^i}{dt^i}\boldsymbol{x}(t) - \boldsymbol{\Omega}(t)\right] = \kappa_2(t)\,\mathbf{A}_0 \sum_{i=0}^{n-1} \frac{t^i}{i!}\frac{d^i}{dt^i}\boldsymbol{x}(0)\,.$$

Combining the results of the above discussion, if $r > m$, and (4) *holds its solution is given by:*

$$\boldsymbol{x}(t) = \kappa_1(t)[\mathbf{Q}_p^1\,\mathrm{e}^{\mathbf{J}_p(t)}\boldsymbol{C} + Q^1\boldsymbol{K}(t) + \mathbf{Q}_\zeta^1 \boldsymbol{z}_\zeta] + \kappa_2(t) \sum_{i=0}^{n-1} \frac{(t)^i}{i!}\frac{d^i}{dt^i}\boldsymbol{x}(0), \quad t \geq 0\,.$$

The dimension of the solution vector space is p.

3 Over–determined electrical machine model

A model commonly utilized in the stability analysis of electric energy systems is the so-called classical electromechanical synchronous machine model. The classical machine model assumes that (i) all stator and rotor dynamics as well as stator losses are null; and (ii) the mechanical and electrical torque can be approximated with the mechanical and electrical power, respectively, as the variations of the rotor angular speed are small. We provide here two possible formulations of the classical machine model as over-determined dynamical system.

Example 1. *These assumptions lead to describing the mechanical (swing) equations as:*

$$\omega_o^{-1}\dot{\delta} = \omega - 1\,, \tag{14}$$

$$M\dot{\omega} = P_{\mathrm{m}} - P_{\mathrm{e}} - D(\omega - 1)\,, \tag{15}$$

where δ, ω, are the rotor's angular position and speed, respectively; ω_o is the synchronous angular speed; M is the mechanical starting time; D is the damping coefficient. P_{m} is the mechanical power; and P_{e} is the electrical power injected by the machine into the grid. In the dq*-axis reference frame, the components of the stator current $\imath_{\mathrm{d}}$, $\imath_{\mathrm{q}}$ are described as:*

$$0 = v_{\mathrm{q}} - e'_{\mathrm{q}} + X'_{\mathrm{d}}\,\imath_{\mathrm{d}}\,, \tag{16}$$

$$0 = v_{\mathrm{d}} - X'_{\mathrm{d}}\,\imath_{\mathrm{q}}\,, \tag{17}$$

where X'_{d} is the d*-axis transient reactance; e'_{q} is the electromotive force "behind the reactance"; v_{d}, v_{q}, are the* d*-axis and* q*-axis components of the machine terminal voltage, respectively. These components can be defined as follows:*

$$0 = v_{\mathrm{d}} - v\sin(\delta - \theta_h)\,, \tag{18}$$

$$0 = v_{\mathrm{q}} - v\cos(\delta - \theta_h)\,, \tag{19}$$

where v, θ are the voltage magnitude and angle, respectively, at the machine terminal bus; The electrical active and reactive power of the machine can be expressed in the dq*-axis reference frame as follows:*

$$0 = v_{\mathrm{d}}i_{\mathrm{d}} + v_{\mathrm{q}}i_{\mathrm{q}} - P_{\mathrm{e}}\,, \tag{20}$$

$$0 = v_{\mathrm{d}}i_{\mathrm{d}} - v_{\mathrm{q}}i_{\mathrm{q}} - Q_{\mathrm{e}}\,. \tag{21}$$

In this model, the mechanical power P_{m} and the electromotive force e'_{q} are considered to be constant. Moreover, the machine is assumed to be connected to a bus with constant voltage and frequency (often called an "infinite" bus), and hence, v_h and θ_h are also constant. Finally, substituting (20) *in* (14) *and assuming that the reactive power injection Q_{e} is an input to the system, the vector of the system variables is defined as:*

$$\boldsymbol{x} = [\delta \quad \omega \quad v_{\mathrm{d}} \quad v_{\mathrm{q}} \quad \imath_{\mathrm{d}} \quad \imath_{\mathrm{q}}]^{\mathrm{T}}\,. \tag{22}$$

The system of (14)-(21) *can be linearized around a valid equilibrium $\boldsymbol{x}^*$. Then, the linearized system can be written in the form of* (1) *as:*

$$\mathbf{E}\,\Delta\dot{\boldsymbol{x}}(t) = \mathbf{A}\,\Delta\boldsymbol{x}(t) + \boldsymbol{\omega}(t)\,, \tag{23}$$

where $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}^*$*;*

$$\mathbf{E} = \begin{bmatrix} \hat{\boldsymbol{E}} \\ \mathbf{0}_{1,6} \end{bmatrix} , \quad \hat{\boldsymbol{E}} = \mathrm{diag}(\omega_o^{-1},\ M,\ 0,\ 0,\ 0,\ 0) ,$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -D & -\imath_{\mathrm{d}}^* & -\imath_{\mathrm{q}}^* & -v_{\mathrm{d}}^* & -v_{\mathrm{q}}^* \\ -v\cos(\delta^* - \theta) & 0 & 1 & 0 & 0 & 0 \\ -v\sin(\delta^* - \theta) & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & X'_{\mathrm{d}} & 0 \\ 0 & 0 & 1 & 0 & 0 & -X'_{\mathrm{d}} \\ 0 & 0 & \imath_{\mathrm{d}}^* & -\imath_{\mathrm{q}}^* & v_{\mathrm{d}}^* & -v_{\mathrm{q}}^* \end{bmatrix} ,$$

and

$$\boldsymbol{\omega}(t) = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \Delta Q_{\mathrm{e}}]^{\mathrm{T}} .$$

Notice that system (23) *is over-determined, since* $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{8\times 7}$.

For illustrative purposes, we consider a specific numerical example. To this aim, we assume $\omega_o = 100\pi$ *rad/s* (50 *Hz system),* $M = 14$ *s,* $D = 3$ *pu*[1], $X'_{\mathrm{d}} = 0.6$ *pu(Ω). Moreover,* $v = 1$ *pu(kV),* $\theta = 0$ *rad,* $e'_{\mathrm{q}} = 1.2$ *pu(kV),* $P_{\mathrm{m}} = 1$ *pu(MW). Then, the equilibrium of the system is:*

$$\boldsymbol{x}^* = [0.524 \;\; 1 \;\; 0.5 \;\; 0.866 \;\; 0.557 \;\; 0.833]^{\mathrm{T}} . \tag{24}$$

The matrices that describe the system are as follows:

$$\mathbf{E} = \begin{bmatrix} 0.003 & 0 & 0 & 0 & 0 & 0 \\ 0 & 14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & -0.557 & -0.833 & -0.5 & -0.866 \\ -1.155 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0.6 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.6 \\ 0 & 0 & 0.557 & -0.833 & 0.5 & -0.87 \end{bmatrix} .$$

[1]per unit system (pu); in power engineering, quantities are often expressed as fractions of defined base units.

The pencil of the system will then be

$$s\mathbf{E}-\mathbf{A} = s\begin{bmatrix} 0.003 & 0 & 0 & 0 & 0 & 0 \\ 0 & 14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & -0.557 & -0.833 & -0.5 & -0.866 \\ -1.155 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.6 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.6 \\ 0 & 0 & 0.557 & -0.833 & 0.5 & -0.87 \end{bmatrix},$$

or, equivalently,

$$s\mathbf{E}-\mathbf{A} = \begin{bmatrix} 0.003s & 0 & 0 & 0 & 0 & 0 \\ 0 & 14s+3 & 0.557 & 0.833 & 0.5 & 0.866 \\ 1.155 & 0 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -0.6 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0.6 \\ 0 & 0 & -0.557 & 0.833 & -0.5 & 0.87 \end{bmatrix},$$

The pencil has the finite eigenvalues $\lambda_1 = -0.107143i + 7.39935$, $\lambda_2 = -0.107143i - 7.39935$, *and infinite eigenvalue of algebraic multiplicity 4 and row minimal index* $\zeta_1 = 0$.

Example 2. *Consider the swing equations* (14)-(15) *of the classical synchronous machine model introduced in example 1. We assume that the machine is connected to a bus* h *with constant voltage* $\bar{v}_h = v_h\angle\theta_h = 1\,\mathrm{pu}\,\angle 0$ *rad. Considering zero damping* ($D = 0$) *as well as constant EMF* $e'_{\mathrm{r,q}}$, *the swing equations (1.113) can be rewritten as follows:*

$$\omega_o^{-1}\dot{\delta} = \omega - 1\,, \tag{25}$$

$$M\dot{\omega} = P_{\mathrm{m}} - P_{\mathrm{e}}^{\max}\sin\,\delta\,, \tag{26}$$

where $P_{\mathrm{e}}^{\max} = \frac{e'_{\mathrm{r,q}}}{X'_{\mathrm{d}}}$. *These two equations now describe a non-dissipative non-linear model which, if perturbed should oscillate forever with oscillations of constant amplitude. However, numerical integration of these equations leads to slightly increasing or decreasing oscillations when solved with certain methods, including the forward and backward Euler methods. This numerical issue can be successfully addressed if the set of differential equations is augmented with a constraint that imposes that the variation of the total free energy (kinetic + potential) of the machine model is zero. Then,* *the numerical integration shows stationary oscillations, as expected. That said, equation (26) can be alternatively expressed as:*

$$M\dot{\omega} = -\frac{\partial \mathcal{V}}{\partial \delta}\,, \tag{27}$$

where $-\frac{\partial \mathcal{V}}{\partial \delta}$ *is the negative gradient of the potential energy function:*

$$\mathcal{V} = -P_{\mathrm{m}}\delta - P_{\mathrm{e}}^{\max}\cos\,\delta\,. \tag{28}$$

Then, the following constraint ensures that the there is no variation of the total free energy of system (25)-(26)*:*

$$\frac{1}{2}M\omega^2 + \mathcal{V} = c\,, \tag{29}$$

where c *is a constant. Substitution of* (28) *in the last equation yields:*

$$\frac{1}{2}M\omega^2 - P_{\mathrm{m}}\delta - P_{\mathrm{e}}^{\max}\cos\,\delta = c\,. \tag{30}$$

The state vector of the system of differential-algebraic equations defined by (25), (26) *and* (30) *is:*

$$\boldsymbol{x} = [\delta,\, \omega]^{\mathrm{T}}\,. \tag{31}$$

Assuming that the mechanical power P_{m} *is an input, linearization of the system around an equilibrium point* $(\boldsymbol{x}^*, \boldsymbol{V}^*)$*, gives:*

$$\begin{aligned} \omega_o^{-1}\Delta\dot{\delta} &= \Delta\omega\,, \\ M\Delta\dot{\omega} &= \Delta P_{\mathrm{m}} - P_{\mathrm{e}}^{\max}\cos\,\delta^*\,\Delta\delta\,, \\ 0 &= M\omega_o\Delta\omega - \Delta\delta - \delta_{\mathrm{r},o}\Delta P_{\mathrm{m}} + P_{\mathrm{e}}^{\max}\sin\,\delta^*\,\Delta\delta\,. \end{aligned} \tag{32}$$

In matrix form, system (32) *reads:*

$$\mathbf{E}\,\Delta\dot{\boldsymbol{x}}(t) = \mathbf{A}\,\Delta\boldsymbol{x}(t) + \boldsymbol{\omega}(t)\,, \tag{33}$$

where

$$\mathbf{E} = \begin{bmatrix} \omega_o^{-1} & 0 \\ 0 & M \\ 0 & 0 \end{bmatrix}\,,\ \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -P_{\mathrm{e}}^{\max}\cos\,\delta^* & 0 \\ -P_{\mathrm{m},o} + P_{\mathrm{e}}^{\max}\sin\,\delta^* & M\omega^* \end{bmatrix}\,,$$

$$\boldsymbol{\omega}(t) = \begin{bmatrix} 0 \\ \Delta P_{\mathrm{m}} \\ -\delta^*\Delta P_{\mathrm{m}} \end{bmatrix}\,.$$

Model (33) *is over-determined, since* $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{3\times 2}$.

We consider a numerical example. In particular, we assume $\omega_o = 100\pi$ *rad/s* (50 *Hz system*), $M = 14$ *s*, $P_{\mathrm{e}}^{\max} = 2$ *pu(MW)*, $P_{\mathrm{m}} = 1$ *pu(MW), Then, the equilibrium of the system state is:*

$$\boldsymbol{x}_o = \left[\frac{\pi}{6}, \quad 1\right]^{\mathrm{T}}. \tag{34}$$

The matrices that describe the system become:

$$\mathbf{E} = \begin{bmatrix} \frac{1}{100\pi} & 0 \\ 0 & 14 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3} & 0 \\ 0 & 14 \end{bmatrix}, \quad \boldsymbol{\omega}(t) = \begin{bmatrix} 0 \\ \Delta P_{\mathrm{m}} \\ -\delta^* \Delta P_{\mathrm{m}} \end{bmatrix}.$$

The pencil of the system will then be

$$s\mathbf{E} - \mathbf{A} = s \begin{bmatrix} \frac{1}{100\pi} & 0 \\ 0 & 14 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\sqrt{3} & 0 \\ 0 & 14 \end{bmatrix},$$

or, equivalently,

$$s\mathbf{E} - \mathbf{A} = \begin{bmatrix} \frac{1}{100\pi}s & -1 \\ \sqrt{3} & 14s \\ 0 & -14 \end{bmatrix},$$

The pencil has the finite eigenvalues $\lambda_1 = i\sqrt{\frac{\sqrt{3}\pi}{14}}$, $\lambda_2 = -i\sqrt{\frac{\sqrt{3}\pi}{14}}$, *and row minimal index* $\zeta_1 = 0$.

4 Illustrative examples

Example 3 (Singular pencil with no solution)**.** *We consider system* (1) *with* $\boldsymbol{\omega}(t) = \mathbf{0}_{4,1}$ *and*

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}.$$

Obviously the system is over–determined, and it's pencil

$$s\mathbf{E} - \mathbf{A} = \begin{bmatrix} s-1 & s-2 & s-2 \\ 0 & s-2 & s-2 \\ s-1 & s-2 & s-2 \\ 0 & s-2 & s-3 \end{bmatrix}$$

is singular. There exists the matrix

$$\hat{\boldsymbol{P}}(s) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3-s & 0 & -s+2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

such that

$$\hat{\boldsymbol{P}}(s)(s\mathbf{E} - \mathbf{A}) = \begin{bmatrix} \hat{\boldsymbol{A}}(s) \\ \mathbf{0}_{1,1} \end{bmatrix},$$

where

$$\hat{\boldsymbol{A}}(s) = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Furthermore, for

$$\hat{\boldsymbol{P}}(s) = \begin{bmatrix} \hat{\boldsymbol{P}}_1(s) \\ \hat{\boldsymbol{P}}_2(s) \end{bmatrix},$$

with

$$\hat{\boldsymbol{P}}_1(s) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3-s & 0 & -s+2 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \qquad \hat{\boldsymbol{P}}_2(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

we have

$$\hat{\boldsymbol{P}}_2(s)\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \neq \mathbf{0}_{1,3}.$$

Hence, (4) *does not hold and* (1) *has no solutions.*

Example 4 (Singular pencil with solutions)**.** *Let*

$$\mathbf{E} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 2 \end{bmatrix}, \qquad \boldsymbol{\omega}(t) = \mathbf{0}_{3,1}.$$

The pencil of the over–dermined system is equal to

$$s\mathbf{E} - \mathbf{A} = \begin{bmatrix} s-1 & s-2 \\ 0 & s-2 \\ s-1 & s-2 \end{bmatrix}.$$

Then

$$\hat{\boldsymbol{P}}(s) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

such that

$$\hat{\boldsymbol{P}}(s)(s\mathbf{E}-\mathbf{A}) = \begin{bmatrix} \hat{\boldsymbol{A}}(s) \\ \mathbf{0}_{1,2} \end{bmatrix}.$$

where

$$\hat{\boldsymbol{A}}(s) = \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}.$$

In addition, for

$$\hat{\boldsymbol{P}}(s) = \begin{bmatrix} \hat{\boldsymbol{P}}_1(s) \\ \hat{\boldsymbol{P}}_2(s) \end{bmatrix},$$

with

$$\hat{\boldsymbol{P}}_1(s) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \hat{\boldsymbol{P}}_2(s) = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix},$$

we have

$$\hat{\boldsymbol{P}}_2(s)\mathbf{E} = \mathbf{0}_{1,2}\,.$$

Hence, (4) *holds and from* (5)*:*

$$\boldsymbol{\Psi}_0(t) = \mathcal{L}^{-1}\{\mathbf{A}^{-1}\hat{\boldsymbol{P}}_1(s)\mathbf{E}\} = \mathcal{L}^{-1}\{\begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix}\},$$

or, equivalently,

$$\boldsymbol{\Psi}_0(t) = \begin{bmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix}.$$

Then for $\boldsymbol{C} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^{\mathrm{T}}$ *we have:*

$$\boldsymbol{x}(t) = \begin{bmatrix} \mathrm{e}^t c_1 \\ \mathrm{e}^{2t} c_2 \end{bmatrix}.$$

Example 5 (Singular pencil with solutions)**.** *We consider now system* (1) *with* $\boldsymbol{\omega}(t) = \mathbf{0}_{5,1}$ *and*

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In this example we will use the spectrum of the pencil $s\mathbf{E}-\mathbf{A}$ *to investigate the solutions of the system. The pencil has the finite eigenvalues* $\lambda_1 = 1$*,*

$\lambda_2 = 2$, and the row minimal indices $\zeta_1 = 0$, $\zeta_2 = 2$. Then since

$$\mathbf{J}_p = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathbf{Q}_p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

by using theorem 2 the solution of (1) is given by:

$$\boldsymbol{x}(t) = \mathbf{Q}_p \, \mathrm{e}^{\mathbf{J}_p t} \boldsymbol{C} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or, equivalently,

$$\boldsymbol{x}(t) = \begin{bmatrix} c_1 \, \mathrm{e}^{t} + c_2 \, \mathrm{e}^{2t} \\ c_2 \, \mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix}.$$

We may now use Corollary 1 to study the uniqueness of solutions of this system. We define the column vector space:

$$colspan \mathbf{Q}_p = < \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} >$$

Since there exist solutions for the system, as proved the uniqueness depends on the initial conditions. Let

$$\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

be the initial conditions of the system. We observe that

$$\boldsymbol{x}(0) \in \mathrm{colspan} \mathbf{Q}_p.$$

Hence from corollary 1 the initial conditions are consistent and the unique solution of (1) is:

$$\boldsymbol{x}(t) = \begin{bmatrix} \mathrm{e}^{t} + \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix}.$$

It is worth noting that the uniqueness is not guaranteed. Had we chosen for example the initial condition:

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

we would have

$$\boldsymbol{x}(0) \notin \mathrm{colspan}\mathbf{Q}_p.$$

Hence from corollary 1 the initial conditions would be non-consistent and the solution of (1) *would be:*

$$\boldsymbol{x}(t) = \kappa_1(t) \begin{bmatrix} c_1\,\mathrm{e}^{t} + c_2\,\mathrm{e}^{2t} \\ c_2\,\mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix} + \kappa_2(t) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad t \geq 0\,.$$

Where $\kappa_1(t)$, $\kappa_2(t)$ defined in remark 8. The dimension of the solution vector space is 2*.*

5 Concluding remarks

The solutions of overdetermined systems in the form of (1) where strictly studied including existence, uniqueness and the formula of its solutions. Results were also extended for higher order systems in the form of (9). We provided several examples and used this type of systems for electrical power system modeling. As a further extension of this article we aim to study the perturbation methods and construct optimization techniques in order to obtain optimal solutions for the case of existence but not uniqueness of solutions for the system.

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