Modal Participation Factors of Algebraic Variables

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***Abstract*— This paper proposes an approach to determine the participation of algebraic variables in power system modes. The approach is based on a new interpretation of the classical participation factors, as well as on the definition of adequate output variables of the system's state-space representation. The paper considers both the linear and generalized eigenvalue problems for the calculation of the participation factors and presents a theorem to cope with eigenvalue multiplicities. An illustrative example on the two-area system, as well as a study on a 1,479-bus dynamic model of the all-Irish transmission system are carried out to support the theory and illustrate the features of the proposed approach.**

***Index Terms*— Small-Signal Stability Analysis (SSSA), Participation Factors (PFs), algebraic variables, Linear Eigenvalue Problem (LEP), Generalized Eigenvalue Problem (GEP).**

I. Introduction

A. Motivation

Classical participation analysis is a tool to measure the coupling of the states of a dynamical system with its modes (eigenvalues of the state matrix). A power system model for angle and voltage stability analysis, however, also includes a variety of algebraic equations and variables, e.g. power flows in network branches, that constrain the system and define its dynamics. To study the impact of the algebraic variables on the system dynamic response is relevant since, very often, the measurements taken on the transmission network and used by local and wide area controllers are modelled as algebraic variables. The focus of this paper is on the participation of algebraic variables in the critical modes of large-scale power systems.

B. Literature Review

Modal participation analysis was first introduced by Pérez-Arriaga *et al.* in [1] and [2]. These studies employed the analytical solution that determines the time response of a linear time-invariant dynamic system and applied initial conditions appropriate to define the relative contribution of a system state in a mode and vice versa. Participation Factors (PFs) were introduced as an approach to Selective Modal Analysis. Nowadays, PFs are widely considered a fundamental tool for power system Small-Signal Stability Analysis (SSSA). They have been also utilized in model reduction [3], as well as in control signal and input placement selection [4].

PFs have been given various interpretations. In terms of eigensensitivities, they represent the sensitivity of an eigenvalue to variations of an element of the state matrix [5]. They can be also viewed as modal energies in the MacFarlane sense [6]. In the state space representation, PFs can be studied as residues of the system transfer function and as joint observabilities/controllabilities of the geometric approach, which play an important role during the design of control systems [7], [8]. The properties of PFs were summarized and extended in [8]. In [9], [10], the authors studied the effect of the uncertainty in the initial conditions in the definition of the PFs. Recent efforts have focused on the definition of PFs for nonlinear systems [11], [12].

Dominant states in lightly damped modes of power systems are typically the synchronous machine rotor angles and speeds. The state variables of poorly tuned controllers, e.g. the Automatic Voltage Regulators (AVRs) and Power System Stabilizers (PSSs), can also show high PFs in critical modes. Nevertheless, measurement units installed on the transmission system buses provide information on the local voltage, frequency and active and reactive power flows, which in angle and voltage stability studies are modelled as algebraic variables [13]. Moreover, these quantities are typically utilized by Flexible AC Transmission System devices as signals for the implementation of various controllers including Power Oscillation Damper (POD) [14].

C. Contributions

This work provides a tool to study how algebraic variables are coupled with power system modes. The focus is on the PFs of bus voltages, frequencies, and power injections; synchronous machines Rate of Change of Frequency (RoCoF); Centre of Inertia (COI) speed of different areas; and any system parameters. However, the formulation provided in the paper is general and can be extended to any nonlinear function of the system states and algebraic variables.

Specific contributions of the paper are as follows:

- A measure for the participation of the algebraic variables and, in general, of any function of the system variables in the system modes, through the definition of appropriate output vectors of the system's state-space representation.
- An alternative interpretation of the classical PFs as eigensensitivities. The proposed interpretation is derived from the partial differentiation of the analytic solution of the linearized power system around an equilibrium point.
- The implementation of modal participation analysis for a power system with eigenvalue multiplicities, modelled as singular system of differential equations, as well as a discussion on how to implement the proposed modal analysis in a large-scale power system.

The paper precisely recognizes that the algebraic variables of a set of Differential Algebraic Equations (DAEs) can be interpreted as functions of the state variables and, in turn, as

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outputs of the state-space representation of the power system model. Until now, algebraic variables were mostly interpreted either as *constraints* and thus eliminated when calculating the state matrix of the system; or as *states with infinitely fast dynamics* and, as such, their PFs to system modes were considered to be null.

The proposed approach is based on the Generalized Eigenvalue Problem (GEP), as opposed to the conventional Linear Eigenvalue Problem (LEP), and fully exploits the sparsity of Jacobian matrices [15]. This allows utilizing solvers for eigenvalue analysis that scale well and are suitable for large real-world systems.

D. Organization

The remainder of the paper is organized as follows. Section II recalls the formulation of the LEP and GEP for power system SSSA. Section III describes the modal participation analysis of a singular power system and introduces a new interpretation of the classical PFs. The proposed approach to measure the participation of algebraic variables in power system modes is presented in Section IV. The case study is discussed in Section V. Finally, conclusions are duly drawn in Section VI.

II. Power System Model for Small Signal Stability Analysis

A. Non-linear power system model and linearization

Power system models for angle and voltage stability analysis are formulated as a set of explicit non-linear DAEs as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\ \mathbf{0}_{m,1} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \ , \end{aligned} \tag{1}$$

where $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$), $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$) are the differential and algebraic equations; $\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{y}$, $\boldsymbol{y} \in \mathbb{R}^m$, are the state and algebraic variables, respectively. For sufficiently small disturbances, (1) can be linearized around a stationary point $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, as follows:

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{f}_x \Delta\boldsymbol{x} + \boldsymbol{f}_y \Delta\boldsymbol{y} \tag{2}$$

$$\mathbf{0}_{m,1} = \boldsymbol{g}_{\boldsymbol{x}} \Delta\boldsymbol{x} + \boldsymbol{g}_y \Delta\boldsymbol{y} \ , \tag{3}$$

where $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0$, $\Delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{y}_0$; $\boldsymbol{f}_x$, $\boldsymbol{f}_y$, $\boldsymbol{g}_x$, $\boldsymbol{g}_y$ are the Jacobian matrices evaluated at $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, i.e. $\boldsymbol{f}_x = \partial\boldsymbol{f}/\partial\boldsymbol{x}\big|_{(\boldsymbol{x}_0,\boldsymbol{y}_0)}$ etc.; and $\mathbf{0}_{i,j}$, $\mathbf{0}_{i,j} \in \mathbb{R}^{i\times j}$ is the zero matrix. Note that the system of (2), (3) is an autonomous linear system, i.e. the elements of $\boldsymbol{f}_x$, $\boldsymbol{f}_y$, $\boldsymbol{g}_x$, $\boldsymbol{g}_y$ are not functions of time t. The objective of SSSA is to study the equilibrium point $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ through the eigenvalue analysis of system (2)-(3).

B. Generalized Eigenvalue Problem

The system (2)-(3) can be written as a set of singular differential equations, as follows:

$$\begin{bmatrix} \boldsymbol{I}_n & \mathbf{0}_{n,m} \\ \mathbf{0}_{m,n} & \mathbf{0}_{m,m} \end{bmatrix} \begin{bmatrix} \Delta\dot{\boldsymbol{x}} \\ \Delta\dot{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_x & \boldsymbol{f}_y \\ \boldsymbol{g}_x & \boldsymbol{g}_y \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{y} \end{bmatrix} \ . \tag{4}$$

Assuming the notation

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{I}_n & \mathbf{0}_{n,m} \\ \mathbf{0}_{m,n} & \mathbf{0}_{m,m} \end{bmatrix} \ , \ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{f}_x & \boldsymbol{f}_y \\ \boldsymbol{g}_x & \boldsymbol{g}_y \end{bmatrix} \ ,$$

where $\boldsymbol{I}_n$, $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$, denotes the identity matrix, we can write:[1]

$$\boldsymbol{B}\dot{\boldsymbol{\xi}} = \boldsymbol{A}\boldsymbol{\xi} \ , \tag{5}$$

$\boldsymbol{\xi} = [\Delta\boldsymbol{x} \ \ \Delta\boldsymbol{y}]^T$. The family of matrices $s\boldsymbol{B} - \boldsymbol{A}$ parametrized by s, $s \in \mathbb{C}$, is called matrix pencil of system (5). In particular, $s\boldsymbol{B} - \boldsymbol{A}$ is a regular matrix pencil, since the matrices $\boldsymbol{B}$, $\boldsymbol{A}$ are square and $\det(s\boldsymbol{B} - \boldsymbol{A}) \not\equiv 0$ [17]. The pencil $s\boldsymbol{B} - \boldsymbol{A}$ has n finite eigenvalues and m infinite eigenvalues. Note that when we refer to infinite eigenvalues, we imply eigenvalues that are at infinity and not infinitely many. If $\boldsymbol{\phi}_i$ and $\boldsymbol{\psi}_i$ are the right and left eigenvectors, respectively, associated with an eigenvalue λ_i, $i = 1, 2, \dots, n+m$, the GEP is described as follows:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{\phi}_i &= \lambda_i \boldsymbol{B}\boldsymbol{\phi}_i \\ \boldsymbol{\psi}_i\boldsymbol{A} &= \lambda_i \boldsymbol{\psi}_i \boldsymbol{B} \ . \end{aligned} \tag{6}$$

Thus, the solution of the GEP consists in calculating the $n+m$ eigenvalues and eigenvectors of $s\boldsymbol{B} - \boldsymbol{A}$.

C. Linear Eigenvalue Problem

The LEP is the conventional eigenvalue problem considered in SSSA of power systems. Algebraic variables are eliminated from system (2)-(3), which leads to a system of ordinary differential equations. Solving (3) for $\Delta\boldsymbol{y}$ yields:

$$\Delta\boldsymbol{y} = -\boldsymbol{g}_y^{-1}\boldsymbol{g}_{\boldsymbol{x}}\Delta\boldsymbol{x} \ , \tag{7}$$

under the assumption that $\boldsymbol{g}_y$ is not singular. Substitution of (7) in (2) leads to the following linear system:

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}_s \Delta\boldsymbol{x} \ , \tag{8}$$

where $\boldsymbol{A}_s = \boldsymbol{f}_x - \boldsymbol{f}_y\boldsymbol{g}_y^{-1}\boldsymbol{g}_x$, $\boldsymbol{A}_s \in \mathbb{R}^{n\times n}$, is the state matrix. The pencil $s\boldsymbol{I}_n - \boldsymbol{A}_s$, has n finite eigenvalues. If $\hat{\boldsymbol{\phi}}_i$ and $\hat{\boldsymbol{\psi}}_i$ are the right and left eigenvectors associated with an eigenvalue $\hat{\lambda}_i$, $i = 1, 2, \dots, n$, the LEP is described as follows:

$$\begin{aligned} \boldsymbol{A}_s\hat{\boldsymbol{\phi}}_i &= \hat{\lambda}_i\hat{\boldsymbol{\phi}}_i \\ \hat{\boldsymbol{\psi}}_i\boldsymbol{A}_s &= \hat{\lambda}_i\hat{\boldsymbol{\psi}}_i \ . \end{aligned} \tag{9}$$

The solution of the LEP consists in calculating the n finite eigenvalues and eigenvectors of $s\boldsymbol{I}_n - \boldsymbol{A}_s$. Note that the finite eigenvalues of (8) and (5) are the same.

III. Participation Factors

A. Conventional Definition

Consider system (8) and the associated LEP described by (9). The PF is defined as the following dimensionless number:

$$p_{k,i} = \hat{\psi}_{i,k}\hat{\phi}_{k,i} \ , \tag{10}$$

[1]In [16], a semi-implicit form of (1) and hence of (2)-(3) is proposed. This formulation leads to a non-diagonal matrix $\boldsymbol{B}$. For simplicity, we do not discuss the semi-implicit form here. However, all results for (5) presented in Section III are valid also for the semi-implicit form given in [16].

where $\hat{\phi}_{k,i}$ is the k-th row element of $\hat{\boldsymbol{\phi}}_i$ and $\hat{\psi}_{i,k}$ is the k-th column element of $\hat{\boldsymbol{\psi}}_i$. $p_{k,i}$ expresses the relative contribution of the k-th state in the i-th mode, and vice versa, under the assumption that all eigenvalues are distinct. The right and left eigenvectors are usually normalized so that the sum of all PFs that correspond to the same mode equals to 1 [1]. However, this is not always the case [18].

The calculated PFs are collected to form a matrix, which is known as the participation matrix. If the right ($\hat{\boldsymbol{\Phi}}$) and left ($\hat{\boldsymbol{\Psi}}$) modal matrices of (8) are defined as $\hat{\boldsymbol{\Phi}} = [\hat{\boldsymbol{\phi}}_1 \ \dots \ \hat{\boldsymbol{\phi}}_n]$, $\hat{\boldsymbol{\Psi}} = [\hat{\boldsymbol{\psi}}_1 \ \dots \ \hat{\boldsymbol{\psi}}_n]^T$, the participation matrix $\boldsymbol{P}$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, can be expressed as:

$$\boldsymbol{P} = \hat{\boldsymbol{\Psi}}^T \circ \hat{\boldsymbol{\Phi}} \ , \tag{11}$$

where $\circ$ denotes the Hadamard product, i.e. the element-wise multiplication, and T denotes the transpose matrix.

B. Participation Factors as Residues

Consider the Single-Input Single-Output (SISO) system state space representation:

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \boldsymbol{A}_s \Delta\,\boldsymbol{x} + \boldsymbol{b}\,u_1 \\ w_1 &= \boldsymbol{c}\,\Delta\boldsymbol{x} \ , \end{aligned} \tag{12}$$

where $\boldsymbol{b}$ is the column vector of the input u_1; $\boldsymbol{c}$ is the row vector of the output w_1. Then, the residue of system (12) transfer function associated with the i-th mode is given by:

$$r_i = \boldsymbol{c}\,\hat{\boldsymbol{\phi}}_i\,\hat{\boldsymbol{\psi}}_i\,\boldsymbol{b} \ . \tag{13}$$

That said, the PF of the k-th state in the i-th mode can be viewed as the residue of system (12) transfer function associated with the i-th mode, when the input is a perturbation in the differential equation that defines $\Delta\dot{x}_k$ and the output is Δx_k. Indeed, if

$$\boldsymbol{c} = \begin{bmatrix} c_1 & \dots & c_k & \dots & c_n \end{bmatrix} = \begin{bmatrix} 0 & \dots & 1 & \dots & 0 \end{bmatrix} \ ,$$

$$\boldsymbol{b}^T = \begin{bmatrix} b_1 & \dots & b_k & \dots & b_n \end{bmatrix}^T = \begin{bmatrix} 0 & \dots & 1 & \dots & 0 \end{bmatrix}^T \ ,$$

equation (13) becomes:

$$r_i = \hat{\psi}_{i,k}\,\hat{\phi}_{k,i} = p_{k,i} \ . \tag{14}$$

In the case of a Multiple-Input Multiple-Output (MIMO) system representation, the PFs appear as the diagonal elements of the emerging residue matrix. The ability to calculate only a subset of all residue elements and acquire an approximate but yet accurate measure of the contribution of system states in system modes (and *vice versa*), features the physical importance and the computational efficiency of the PFs.

C. Participation Analysis of Large-Scale Systems

For a system of small to medium size, one can efficiently compute all the right and left eigenvectors and determine the participation matrix from (11). However, a property of the state matrix $\boldsymbol{A}_s$ is that it is dense. The standard algorithm for the solution of the eigenvalue problem for dense matrices is QR factorization, e.g. with LAPACK [19], which calculates all eigenvalues and eigenvectors. The QR algorithm has computational complexity $O(n^3)$ and therefore, it is not practical for large systems. For example, with the current typical processing capacities, solution of the LEP for a dynamic model of the ENTSO-E system that includes about 40k state variables (thus leading to a dense $\boldsymbol{A}_s$ with about 1.6M elements) cannot be solved with LAPACK and a memory error is returned [15].

On the other hand, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the GEP are sparse. A commonly utilized algorithm that is appropriate for large sparse matrices is Arnoldi iteration and its variants, e.g. with ARPACK [20] coupled with an efficient sparse factorization solver such as KLU [21]). A library that has also shown promising results is Z-PARES [22], which allows determining the eigenvalues that lie in a domain of the complex plane defined by the user. Apart from exploiting sparsity, an advantage of these libraries is that they allow efficiently calculating only a subset of the solution, i.e. the most critical eigenvalues and associated eigenvectors. For this reason, in this paper, we consider the modal participation analysis of the GEP, i.e. the singular system (5).

As already mentioned, the main assumption of classical participation analysis is that all eigenvalues are distinct. However, it is common in the simulation of dynamical system models that some eigenvalues are repeated. For small size systems, it is feasible to perturb some parameters and avoid multiplicities. But this is impractical for a real-world size system.

This paper provides the implementation of modal participation analysis for a power system with multiple eigenvalues. The formulation of the PFs for the special cases of distinct eigenvalues and multiple eigenvalues with algebraic multiplicity equal to the geometric one is also extracted.

In the remainder of this subsection, we first derive the analytical solution of a singular system with multiple eigenvalues. For this system, we then propose a new interpretation of the PFs, which is valid for both GEP and LEP. Rewriting (5) as:

$$\boldsymbol{B}\dot{\boldsymbol{\xi}}(t) = \boldsymbol{A}\boldsymbol{\xi}(t) \ , \tag{15}$$

where $t \in [0, \infty)$, we can propose the following theorem.

Theorem 1. Consider the system (15) with initial conditions $\boldsymbol{\xi}(0)$ and regular pencil $s\boldsymbol{B} - \boldsymbol{A}$. Let λ_i, $i = 1, 2, ..., \nu$ be a finite eigenvalue of the pencil, where ν corresponds to the number of Jordan blocks. Let also n_i be the rank of the corresponding Jordan block. We have $\sum_{i=1}^{\nu} n_i = n$, $\nu \le n$. Then, the general solution of (5) can be written as follows:

$$\boldsymbol{\xi}(t) = \sum_{i=1}^{\nu} e^{\lambda_i t} \sum_{j=1}^{n_i} \big(\sum_{\sigma=1}^{j} t^{\sigma-1} \boldsymbol{\psi}_i^{(j-\sigma+1)} \boldsymbol{B}\boldsymbol{\xi}(0) \big) \boldsymbol{\phi}_i^{(j)}, \tag{16}$$

where $\boldsymbol{\phi}_i^{(j)}$, $\boldsymbol{\psi}_i^{(j)}$ denote the j-th (generalized) right, left eigenvectors corresponding to the eigenvalue λ_i, respectively.

The proof of (16) is given in the appendix. From the proof of Theorem 3.1, we arrive at the following proposition.

Proposition 1. Consider the system (5) with a regular pencil. Let $\xi_k(t)$ be the k-th element of $\boldsymbol{\xi}(t)$, with $k \le n$, i.e. ξ_k is a state variable. Then the PF of ξ_k in the finite mode λ_i is given by the sensitivity:

$$p_{k,i} = \frac{\partial \xi_k(t)}{\partial e^{\lambda_i t}} \ . \tag{17}$$

The proof of (17) is given in the appendix.

The following comments are relevant:

(a) Since only the finite eigenvalues appear in (16), the participation matrix of system (5) has dimensions $(n+m)\times\nu$. In order to determine the PFs associated with the infinite modes and obtain the full matrix, one can apply a special Möbius transform, i.e. the dual transform, in (5) and employ the eigenvectors associated to the zero eigenvalue of the dual pencil $\hat{s}\boldsymbol{A}-\boldsymbol{B}$, where $\hat{s}=1/s$. Nevertheless, infinite modes are not dynamics of interest in SSSA, thus the complete analysis, even though it is interesting from a mathematical viewpoint, it is not relevant for dynamical system studies. For more information, the interested reader can refer to [23].

(b) Consider (6) for $n<k\leq n+m$, i.e. ξ_k is an algebraic variable. Then $\boldsymbol{\psi}_i\boldsymbol{B}\boldsymbol{\xi}(0)=0$. The m rightmost columns of $\boldsymbol{B}$ which contain only zero elements, impose that the PFs of the algebraic variables in the system finite modes are found to be null. This is a consequence of the fact that the coefficients of the first derivatives of the algebraic variables are zero, which implies that the algebraic variables introduce only infinite modes to the system. Nevertheless, the algebraic variables constrain the system and, in this sense, do participate in the system finite modes. This will be further discussed in the next section, where the participation of the algebraic variables is seen through the PFs of the system states.

(c) As already discussed, the GEP (6) is preferable for large networks and allows determining only the part of the participation matrix that is associated with the most critical modes. Let κ, $\kappa\leq\nu$, be the number of the calculated finite eigenvalues. If the corresponding right and left modal matrices are denoted with $\boldsymbol{\Phi}_\kappa$ and $\boldsymbol{\Psi}_\kappa$, respectively, then using (17) the (critical) participation matrix can be always expressed as:

$$\boldsymbol{P}_\kappa=\boldsymbol{\Psi}_\kappa^T\circ(\boldsymbol{B}\boldsymbol{\Phi}_\kappa)=\begin{bmatrix}\boldsymbol{P}_x\\ \boldsymbol{0}_{m,\kappa}\end{bmatrix}\ , \tag{18}$$

where $\boldsymbol{P}_x\subset\boldsymbol{P}$, $\boldsymbol{P}_x\in\mathbb{C}^{n\times\kappa}$. The matrix $\boldsymbol{P}_x$ contains all the information on the dynamics of interest and is the matrix that is utilized in the remainder of the paper.

IV. Participation factors of algebraic variables

In this section, we introduce an approach to measure the participation of algebraic variables in power system modes. These can be algebraic variables included in the DAE system model, or, in general, any algebraic outputs that is defined as a function of the states and algebraic variables of the DAE system.

Let us define the output vector $\boldsymbol{w}$, $\boldsymbol{w}\in\mathbb{R}^q$ such that:

$$\boldsymbol{w}=\boldsymbol{h}(\boldsymbol{x},\boldsymbol{y})\ ,$$

where $\boldsymbol{h}$ ($\boldsymbol{h}:\mathbb{R}^{n+m}\rightarrow\mathbb{R}^q$) is a nonlinear function of $\boldsymbol{x}$, $\boldsymbol{y}$. Then differentiation around $(\boldsymbol{x}_0,\boldsymbol{y}_0)$ yields:

$$\Delta\boldsymbol{w}=\boldsymbol{h}_x\Delta\boldsymbol{x}+\boldsymbol{h}_y\Delta\boldsymbol{y}\ , \tag{19}$$

where $\boldsymbol{h}_x=\partial\boldsymbol{h}/\partial\boldsymbol{x}\big|_{(\boldsymbol{x}_0,\boldsymbol{y}_0)}$, $\boldsymbol{h}_y=\partial\boldsymbol{h}/\partial\boldsymbol{y}\big|_{(\boldsymbol{x}_0,\boldsymbol{y}_0)}$. Substitution of (7) to the last equation gives:

$$\Delta\boldsymbol{w}=\boldsymbol{C}\Delta\boldsymbol{x}\ , \tag{20}$$

where $\boldsymbol{C}=\boldsymbol{h}_x-\boldsymbol{h}_y\boldsymbol{g}_y^{-1}\boldsymbol{g}_{\boldsymbol{x}}$, $\boldsymbol{C}\in\mathbb{R}^{q\times n}$, is the output matrix.

Let Δw_μ be the μ-th system output. We propose the following expression as the PF of Δw_μ in the mode λ_i:

$$\pi_{\mu,i}=\frac{\partial\Delta w_\mu}{\partial e^{\lambda_i t}}\ . \tag{21}$$

From the state-space viewpoint, $\pi_{\mu,i}$ expresses the residue (or the joint observability/controllability) of the i-th mode, when the input is, exactly as it holds for $p_{k,i}$, a perturbation in the differential equation that defines $\Delta\dot{x}_k$. The output however is Δw_μ, which can be, in principle, any function of the system state variables. The fact that the perturbation that leads to (17) and (21) is the same, is also the reason that $\pi_{\mu,i}$ is called PF.

Proposition 2. Let the PF $\pi_{\mu,i}$ be the μ-th row, i-th column element of the participation matrix $\boldsymbol{\Pi}_{(\boldsymbol{w})}$. Then:

$$\boldsymbol{\Pi}_{(\boldsymbol{w})}=\boldsymbol{C}\boldsymbol{P}_x\ . \tag{22}$$

Proof. Let $\boldsymbol{c}_\mu=\begin{bmatrix}c_{\mu,1} & \dots & c_{\mu,n}\end{bmatrix}$ be the μ-th row of $\boldsymbol{C}$. Then, we have for Δw_μ:

$$\Delta w_\mu=\boldsymbol{c_\mu}\Delta\boldsymbol{x}=c_{\mu,1}\Delta x_1+c_{\mu,2}\Delta x_2+\dots+c_{\mu,n}\Delta x_n\ .$$

Partial differentiation over $e^{\lambda_i t}$ leads to:

$$\begin{aligned}\frac{\partial\Delta w_\mu}{\partial e^{\lambda_i t}}&=c_{\mu,1}\frac{\partial\Delta x_1}{\partial e^{\lambda_i t}}+c_{\mu,2}\frac{\partial\Delta x_2}{\partial e^{\lambda_i t}}+\dots+c_{\mu,n}\frac{\partial\Delta x_n}{\partial e^{\lambda_i t}}+\\ &\quad+\frac{\partial c_{\mu,1}}{\partial e^{\lambda_i t}}\Delta x_1+\frac{\partial c_{\mu,2}}{\partial e^{\lambda_i t}}\Delta x_2+\dots+\frac{\partial c_{\mu,n}}{\partial e^{\lambda_i t}}\Delta x_n\\ \Rightarrow\pi_{\mu,i}&=c_{\mu,1}p_{1,i}+c_{\mu 2}p_{2,i}+\dots+c_{\mu n}p_{n,i}\ ,\end{aligned}$$

where $\frac{\partial c_{\mu,1}}{\partial e^{\lambda_i t}}=\frac{\partial c_{\mu,2}}{\partial e^{\lambda_i t}}=\ldots=\frac{\partial c_{\mu,n}}{\partial e^{\lambda_i t}}=0$, since the elements of $\boldsymbol{C}$ do not depend on functions of t. By applying the same steps for all outputs and representing in matrix form, we arrive at (22).

The proof is completed.

The main feature of (22) is that it allows defining the participation matrix not only of the algebraic variables of the DAE, but also of any defined output vector that is a function of the system state and algebraic variables. One has only to specify the gradients $\boldsymbol{h}_x$ and $\boldsymbol{h}_y$ at the operating point, and then calculate the output matrix $\boldsymbol{C}$. The proposed participation matrix $\boldsymbol{\Pi}_{(\boldsymbol{w})}$ provides meaningful information for the system coupling that, to the best of the authors' knowledge, has not been exploited in the literature.

Remark 2. We enumerate the following important special cases for the participation matrix of (22):

(a) *State variables:* If $\boldsymbol{w}=\boldsymbol{x}$, the gradients in (19) become $\boldsymbol{h}_x=\boldsymbol{I}_n$, $\boldsymbol{h}_y=\boldsymbol{0}$. The output matrix is $\boldsymbol{C}=\boldsymbol{I}_n$ and hence the participation matrix of the system states is, as to be expected:

$$\boldsymbol{\Pi}_{(\boldsymbol{x})}=\boldsymbol{P}_x\ . \tag{23}$$

(b) *Algebraic variables:* If $\boldsymbol{w} = \boldsymbol{y}$, the gradients in (19) become $\boldsymbol{h}_x = \boldsymbol{0}$, $\boldsymbol{h}_y = \boldsymbol{I}_m$. The output matrix is $\boldsymbol{C} = -\boldsymbol{g}_y^{-1}\boldsymbol{g}_{\boldsymbol{x}}$. Thus:

$$\boldsymbol{\Pi}_{(\boldsymbol{y})} = -\boldsymbol{g}_y^{-1}\boldsymbol{g}_{\boldsymbol{x}}\boldsymbol{P}_x \ , \tag{24}$$

which is the participation matrix of the algebraic variables in system modes included in the DAE model.

(c) *Rates of change of state variables:* If we have the output $\boldsymbol{w} = \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$, we obtain the participation matrix of the derivatives of the state variables in system modes. The Rate of Change of Frequency (RoCoF) of the synchronous machines ($\dot{\boldsymbol{\omega}}_{\rm G}$) is a relevant case. The gradients in (19) become $\boldsymbol{h}_x = \boldsymbol{f}_x$, $\boldsymbol{h}_y = \boldsymbol{f}_y$. The output matrix is $\boldsymbol{C} = \boldsymbol{A}$. Thus:

$$\boldsymbol{\Pi}_{(\dot{\boldsymbol{x}})} = \boldsymbol{A}\boldsymbol{P}_x \ . \tag{25}$$

(d) *Parameters:* Finally, consider the scalar output $w = \eta$, where η is a defined parameter. If η appears only in the j-th algebraic equation $0 = g^j(\boldsymbol{x}, \boldsymbol{y}, \eta)$, then we can obtain the participation vector of η in the system modes. Linearization of the j-th algebraic equation around the operating point yields:

$$0 = \boldsymbol{g}_x^j \Delta\boldsymbol{x} + \boldsymbol{g}_y^j \Delta\boldsymbol{y} + g_\eta^j \Delta\eta \ , \tag{26}$$

where $\boldsymbol{g}_x^j \in \mathbb{R}^{1\times n}$, $\boldsymbol{g}_y^j \in \mathbb{R}^{1\times m}$ and $g_\eta^j \in \mathbb{R}_{\neq 0}$. Solving (26) for $\Delta\eta$ and comparing with (19), we obtain that $\boldsymbol{h}_x = -\boldsymbol{g}_x^j/g_\eta^j$ and $\boldsymbol{h}_y = -\boldsymbol{g}_y^j/g_\eta^j$. The participation vector is obtained from (22) for $\boldsymbol{C} = (-\boldsymbol{g}_x^j + \boldsymbol{g}_y^j\boldsymbol{g}_y^{-1}\boldsymbol{g}_{\boldsymbol{x}})/g_\eta^j$.

Notice, finally, that once the eigenvalue analysis is completed and the modal matrices are known, calculating the proposed participation matrices involves few matrix multiplications. From the computational burden viewpoint, the cost of calculating the PFs is marginal compared to the eigenvalue analysis.

V. Case Studies

In this case study, we present two practical applications of the proposed approach and show how defining PFs of algebraic variables in system modes can help design more effective and robust controllers. In particular, Section V-A is based on the well-known two-area system [24] and shows how the calculation of PFs can help select the most effective algebraic variable to be measured to damp interarea oscillations. Section V-B utilises a realistic detailed model of the all-island Irish power system and shows how PFs can help define the impact of a given system mode on the network. This second case study also serves to discuss the robustness and the scalability of our approach.

All simulations are carried using the Python-based software tool DOME [25]. The version of DOME employed here is based on Fedora Linux 28, Python 3.6.8, CVXOPT 1.2.2 [26], KLU 1.3.8, and MAGMA 2.2.0 [27]. The hardware consists of two 20-core 2.2 GHz Intel Xeon CPUs which are employed for general matrix factorization, and one NVIDIA Tesla K80 GPU, which is employed for SSSA.

A. Two-Area System

The two-area system is depicted in Fig. 1. It comprises two areas connected through a relatively weak tie; eleven buses and four synchronous machines. Each generator is equipped with an AVR of type IEEE DC-1 and a turbine governor. The system feeds two loads connected at buses B7 and B9 and which are modelled as constant active and reactive power consumptions.

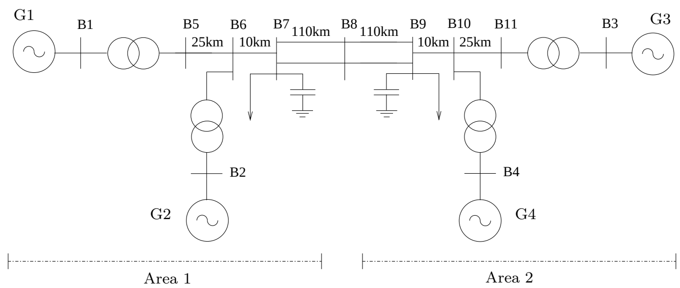

Fig. 1: Two-area four-machine system.

The system model has 52 state variables. For a system with this dynamic order, we can efficiently calculate the state matrix $\boldsymbol{A}_s$. The most critical modes and the mostly participating states in these modes are presented in Table I. Area 1 presents a critical local mode $-0.599 \pm j6.604$ with natural frequency 1.06 Hz and dominant state the rotor speed ω_{G2}. Area 2 presents a critical local mode as well, which is $-0.514 \pm j6.843$ with natural frequency 1.09 Hz and dominant state the rotor speed ω_{G4}. For these modes, the damping ratio is $> 5\%$. Finally, the most lightly damped mode is $-0.096 \pm j3.581$, which is an interarea mode with natural frequency 0.57 Hz. The mostly participating state in the interarea mode is the rotor speed ω_{G3}.

TABLE I: Two-area system critical modes

Mode	Freq. (Hz)	Damping (%)	x-dom.	$\|p\|_{max}$
$-0.096 \pm j3.581$	0.57	2.67	ω_{G3}	0.1696
$-0.514 \pm j6.843$	1.09	7.50	ω_{G4}	0.2945
$-0.599 \pm j6.604$	1.06	9.04	ω_{G2}	0.2530

The participation matrix of the algebraic variables for these modes is calculated from (24). Note that in this section, each $\pi_{\mu,i}$ is divided over the Euclidean norm of the respective output $\boldsymbol{c}_{\boldsymbol{\mu}}$, so that the results are normalized and comparable according to the geometric approach. Of course, since the PFs are a relative measure, one may apply any further normalization, e.g. the maximum or the sum of the values to be equal to 1.

We carry a simple test, to show how the proposed PFs of the algebraic variables are linked to their sensitivities in eigenvalue changes. We impose a perturbation in the active power and voltage of the PV buses B1, B4, so that the most critical mode $-0.096 \pm j3.581$ changes by $|d\lambda| = 3 \cdot 10^{-5}$. The calculated eigensensitivities $|d\Delta y_k|/|d\lambda|$ are compared with the PFs of the algebraic variables P_1, P_4, v_1, v_4, in Table II. As expected, a highly participating variable in a mode indicates that this mode is sensitive to small variations of this variable.

TABLE II: Sensitivity test for the interarea mode, $|d\lambda| = 3 \cdot 10^{-5}$.

y_k (pu)	$\pi_{k,i}$	$\lvert d\Delta y_k\rvert$ (pu)	$\frac{\lvert d\Delta y_k\rvert}{\lvert d\lambda\rvert}$
$P_1 = 5.88$	0.3642	$0.43 \cdot 10^{-3}$	14.42
$P_4 = 7.00$	0.9766	$0.54 \cdot 10^{-3}$	18.00
$v_1 = 1.03$	0.0036	$0.27 \cdot 10^{-4}$	0.90
$v_4 = 1.01$	0.0028	$0.11 \cdot 10^{-4}$	0.35

For illustration, we now consider the simple example of finding the participation vector of one system parameter. Let P_7 be the active power consumption of the load connected at B7. Since P_7 is also the active power injection at B7, we can write the following algebraic equation (see Fig. 1):

$$
\begin{aligned}
0 &= v_7 v_6 (G_{76}\cos(\theta_7 - \theta_6) + B_{76}\sin(\theta_7 - \theta_6)) \\
&\quad + v_7 v_8 (G_{78}\cos(\theta_7 - \theta_8) + B_{78}\sin(\theta_7 - \theta_8)) - P_7 \\
&= g(v_6, v_7, v_8, \theta_6, \theta_7, \theta_8, P_7) \ .
\end{aligned}
$$

Linearization and solving for ΔP_7 yields:

$$
\begin{aligned}
\Delta P_7 = \big(&\frac{\partial g}{\partial v_6}\Delta v_6 + \frac{\partial g}{\partial v_7}\Delta v_7 + \frac{\partial g}{\partial v_8}\Delta v_8 + \frac{\partial g}{\partial \theta_6}\Delta\theta_6 \\
&+ \frac{\partial g}{\partial \theta_7}\Delta\theta_7 + \frac{\partial g}{\partial \theta_8}\Delta\theta_8\big) \ ,
\end{aligned}
$$

where, the gradients are evaluated at $(v_6^0, v_7^0, v_8^0, \theta_6^0, \theta_7^0, \theta_8^0)$; we substituted $\partial g/\partial P_7 = -1$. Therefore, we obtain that $\boldsymbol{h}_x = \boldsymbol{0}$ and $\boldsymbol{h}_y$ is the $1 \times m$ row vector which contains the gradients calculated above in the indexes of $v_6, v_7, v_8, \theta_6, \theta_7, \theta_8$; all other elements of $\boldsymbol{h}_y$ are zero. The output matrix $\boldsymbol{C}$ is $\boldsymbol{C} = -\boldsymbol{h}_y \boldsymbol{g}_y^{-1} \boldsymbol{g}_{\boldsymbol{x}}$, $\boldsymbol{C} \in \mathbb{R}^{1\times n}$. The resulting participation matrix is given by (22).

TABLE III: Participation factors, two-area system.

Mode	$-0.096 \pm j3.581$		$-0.514 \pm j6.843$		$-0.599 \pm j6.604$	
Output	Dom.	$\lvert\pi\rvert$	Dom.	$\lvert\pi\rvert$	Dom.	$\lvert\pi\rvert$
$\boldsymbol{v}_{\mathrm{B}}$	v_{11}	0.0192	v_8	0.0375	v_7	0.0345
$\boldsymbol{\theta}_{\mathrm{B}}$	θ_8	0.1429	θ_4	0.2385	θ_6	0.2250
$\boldsymbol{\omega}_{\mathrm{B}}$	ω_8	0.2065	ω_{10}	0.3247	ω_6	0.3113
$\boldsymbol{P}_{\mathrm{B}}$	P_6	0.1447	P_{10}	0.2518	P_6	0.2719
$\boldsymbol{Q}_{\mathrm{B}}$	Q_{11}	0.0258	Q_8	0.0544	Q_{10}	0.0631
$\dot{\boldsymbol{\omega}}_{\mathrm{G}}$	$\dot{\omega}_{G4}$	0.0401	$\dot{\omega}_{G4}$	0.0917	$\dot{\omega}_{G2}$	0.0539
$\boldsymbol{\omega}_{\mathrm{coi}}$	$\omega_{coi,2}$	0.1700	$\omega_{coi,2}$	0.3151	$\omega_{coi,1}$	0.3137

The active ($\boldsymbol{P}_{\mathrm{B}}$) and reactive ($\boldsymbol{Q}_{\mathrm{B}}$) power injections on all system buses, as well as the COI speeds ($\boldsymbol{\omega}_{\mathrm{coi}}$) of the two areas are defined as outputs and their PFs are obtained from (22). Correspondingly, the system bus voltages ($\boldsymbol{v}_{\mathrm{B}}$), angles ($\boldsymbol{\theta}_{\mathrm{B}}$) and frequencies ($\boldsymbol{\omega}_{\mathrm{B}}$) are included in the algebraic variables of the DAEs. Thus, their PFs are determined from (24). With this aim, ideal frequency estimations of the system buses are obtained by employing the Frequency Divider Formula (FDF) proposed in [28]. The formulation of the FDF in per units is as follows:

$$
\boldsymbol{B}_{\mathrm{BB}}\Delta\boldsymbol{\omega}_{\mathrm{B}} = -\boldsymbol{B}_{\mathrm{BG}}\Delta\boldsymbol{\omega}_{\mathrm{G}} \ ,
$$

where $\Delta\boldsymbol{\omega}_{\mathrm{B}}$ are the estimated bus frequency deviations with respect to the reference synchronous speed; $\Delta\boldsymbol{\omega}_{\mathrm{G}}$ are the synchronous machines rotor speed deviations; and $\boldsymbol{B}_{\mathrm{BB}}$, $\boldsymbol{B}_{\mathrm{BG}}$ are system susceptance matrices that include the internal reactances of the synchronous machines. Finally, the PFs of the RoCoF of the synchronous machines ($\dot{\boldsymbol{\omega}}_{\mathrm{G}}$) are determined from (25).

The mostly participating of the above variables in the system critical modes are summarized in Table III. We observe that the bus voltages, the reactive power injections and the RoCoF have a low participation in the system critical modes. Mostly participating in the interarea mode is the bus frequency ω_8. Similarly, the bus frequency ω_{10} is the one mostly participating in the local mode of Area 2. Finally, the COI speed of Area 1 ($\omega_{coi,1}$) is the one mostly participating in $-0.599 \pm j6.604$, which is a local mode of this area.

Finally, we show how the calculated PFs can be utilized to improve the dynamic behaviour of the system. As already discussed, the critical mode of the system is the interarea mode and the mostly participating variable (Table III) is the bus frequency ω_8. We install a Static Var Compensator (SVC) at B8 with a POD loop [14]. The POD input signal is ω_8. The POD output is considered as an additional input to the SVC voltage reference algebraic equation. The results are summarized in Table IV. Eigenvalue analysis shows that, after the addition of the controller, the system is stable and no mode is poorly damped.

TABLE IV: Impact of SVC-POD installation in the critical mode.

SVC-POD	Mode	Damping (%)
No	$-0.096 \pm j3.581$	2.67
Yes	$-0.256 \pm j3.562$	7.16

B. All-Island Irish Transmission System

In this section we consider a real-world model of the all-island Irish power system. The topology and the steady-state operation data of the system have been provided by the Irish transmission system operator, EirGrid Group, whereas the dynamic data have been defined based on our knowledge about the technology of the generators and the controllers. The system consists of 1,479 buses, B1 . . . B1479, 796 lines, 1055 transformers, 245 loads, 22 synchronous machines G1 . . . G22, with AVRs and turbine governors, 6 PSSs and 176 wind power plants.

The dynamic order of the system is 1,480. The eigenvalue analysis shows that the system is stable when subject to small disturbances. The system presents both local machine modes and intermachine modes. Recall that, a local machine mode refers to a single machine oscillating against the rest of the system. On the other hand, an intermachine mode refers to a group of machines of the same area oscillating against each other [24]. In the remainder of this section we show two modes with different damping ratios and natural frequencies. The examined modes are summarized in Table V.

Mode 1 has eigenvalue $-0.586 \pm j7.248$, with natural frequency 1.16 Hz and damping ratio 8.06%. The dominant states in this mode are the rotor angle and speed of the synchronous machine G16. The PFs of these states sum to 0.8912. The mode is local with G16 oscillating against the

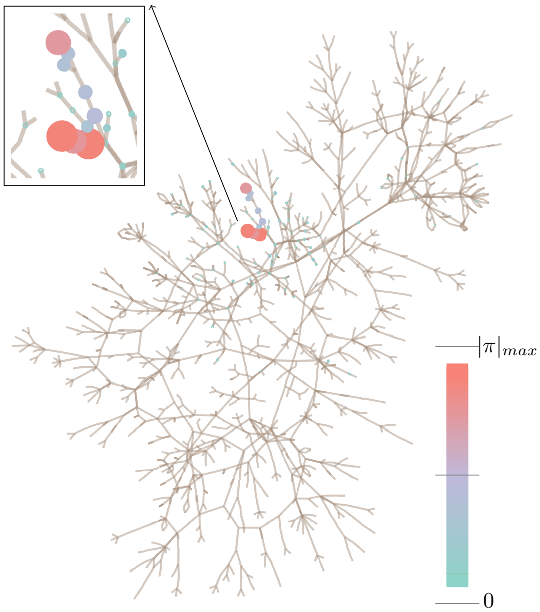

Fig. 2: Participation of bus active power injections in Mode 1 for the all-island Irish system.

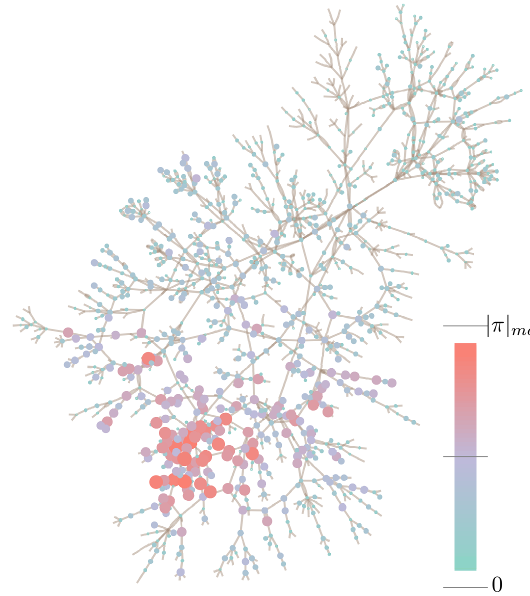

Fig. 3: Participation of bus active power injections in Mode 2 for the all-island Irish system.

rest of the system. Mode 2 has eigenvalue $-0.722 \pm j4.618$, with frequency 0.74 Hz and damping ratio 15.44 %. The mostly participating states are the rotor speed and angle of the synchronous machine G2. The corresponding PFs sum to 0.5755. The natural frequency and the distribution of the PFs indicate that this is an intermachine mode [24].

The Python module *graph-tool* [29] is utilized to generate a graph of the studied network. The resulting graph has 1,479 vertices, which correspond to the system buses and 1,851 edges, which correspond to lines and transformers. Note that the coordinates of the graph vertices and edges do not represent the actual geography of the system. For the considered modes, we calculate the participation matrices of the bus active power injections. Then, the sizes and the colors of the graph vertices are adjusted with respect to the magnitude of the calculated PFs.

The generated graph with the PFs of all bus active power injections in the local Mode 1 is illustrated in Fig. 2. The mostly participating active power injection is the one of the bus B552, that is adjacent to the machine G16, with $|\pi|_{max} = 0.3218$. The PFs of all bus active power injections in the intermachine Mode 2 is illustrated in Fig. 3. The mostly participating active power injection is the one of the bus B1405, that is close to the synchronous machine G2, with $|\pi|_{max} = 0.2508$. Figure 3 shows that the lower frequency oscillations spread over the power system. In fact, there are several buses in a large area that have a high participation in the intermachine mode.

TABLE V: Examined modes, Irish system.

Mode	Mode 1		Mode 2	
Eigenvalue	$-0.586 \pm j7.248$		$-0.722 \pm j4.618$	
Frequency (Hz)	1.16		0.74	
Damping (%)	8.06		15.44	
Type	Local		Intermachine	
Dominant States	State	$\lvert p \rvert_{\max}$	State	$\lvert p \rvert_{\max}$
1st	δ_{G16}	0.4456	ω_{G2}	0.2883
2nd	ω_{G16}	0.4456	δ_{G2}	0.2872

We complete this case study with a discussion on the computational burden of the proposed approach to calculate PFs. Table VI shows a comparison of the LEP solved with LAPACK and the GEP solved with ARPACK, for the same Irish system. Regarding the participation analysis, once the eigenvalue analysis is completed, the cost of calculating the proposed PFs is negligible.

TABLE VI: Eigenanalysis computational burden, Irish system.

Problem	LEP	GEP
Pencil	$s\boldsymbol{I}_n - \boldsymbol{A}_s$	$s\boldsymbol{B} - \boldsymbol{A}$
Solver	Schur method	Arnoldi iteration
Library	LAPACK	ARPACK
Size	1480×1480	8578×8578
Eigenvalues	All	20 LR[2]
Eigen-analysis completed in [s]	14.50	0.68
$\boldsymbol{P}_{(x)}$ computed in [s]	0.08	0.39

Based on the results of Table VI, the following remarks are relevant:

- LEP (LAPACK): All finite eigenvalues and eigenvectors all calculated. The matrix multiplications and transpositions required to compute $\boldsymbol{P}_x$ are completed in 0.08 s. The calculation of the proposed matrix $\boldsymbol{\Pi}_{(w)}$ requires multiplication with matrix $\boldsymbol{C}$. The multiplication time depends on the size of matrix $\boldsymbol{C}$, but for most of the cases is marginal.
- GEP (ARPACK): In this case, only the 20 eigenvalues with the largest real value are calculated. For the calculation of the eigenvalues, a Cayley transform has been

[2]LR: Largest real value.

considered with shift and anti-shift parameters $\sigma = 0.01$, $\kappa = -0.05$, respectively. Since it takes advantage of the sparsity of the matrix pencil, the eigenvalue analysis is completed much faster. Note that solving the GEP with a sparse solver is the only solution possible if a very large dynamic system is to be considered [15]. The matrix multiplications and transpositions required to compute $\boldsymbol{P}_x$ are completed in 0.39 s.

In conclusion, the proposed approach allows exploiting the sparsity of the GEP matrix pencil and can lead to a significant speedup, provided that a proper eigenvalue solver is employed.

VI. Conclusions

The paper proposes a systematic analytical approach to quantify the participation of the algebraic variables of a power system model, and in general of any function of the system variables in the system modes, through the definition of output vectors of the system's state-space formulation. The proposed approach, which describes an alternative interpretation of the PFs as eigensensitivities, provides a high flexibility, since it allows determining PFs of states, algebraic variables, rates of change of system variables, as well as of system parameters. In future work, we aim at utilizing the approach proposed here to design robust wide-area controllers.

Appendix

This appendix provides the proofs of (16) and (17).

Proof of Theorem 1. Since $s\boldsymbol{B} - \boldsymbol{A}$ is regular, there exist non-singular matrices $\boldsymbol{\Psi}$, $\boldsymbol{\Phi} \in \mathbb{C}^{(n+m)\times(n+m)}$ such that

$$\begin{aligned} \boldsymbol{\Psi}\boldsymbol{B}\boldsymbol{\Phi} &= \boldsymbol{I}_n \oplus \boldsymbol{H}_m \\ \boldsymbol{\Psi}\boldsymbol{A}\boldsymbol{\Phi} &= \boldsymbol{J}_n \oplus \boldsymbol{I}_m \ , \end{aligned} \tag{27}$$

where $\boldsymbol{J}_n$, $\boldsymbol{J}_n \in \mathbb{C}^{n\times n}$ is the Jordan matrix related to the finite eigenvalues, $\boldsymbol{H}_m \in \mathbb{C}^{m\times m}$ is a nilpotent matrix constructed by using the algebraic multiplicity of the infinite eigenvalue. By substituting the transformation

$$\boldsymbol{\xi}(t) = \boldsymbol{\Phi}\boldsymbol{z}(t) \tag{28}$$

into (15), and by multiplying by $\boldsymbol{\Psi}$ we obtain

$$\boldsymbol{\Psi}\boldsymbol{B}\boldsymbol{\Phi}\dot{\boldsymbol{z}}(t) = \boldsymbol{\Psi}\boldsymbol{A}\boldsymbol{\Phi}\boldsymbol{z}(t) \ . \tag{29}$$

Let $\boldsymbol{\Phi}_n$, $\boldsymbol{\Phi}_m$ be the matrices that contain all right eigenvectors of the finite, and infinite eigenvalues respectively. Then by setting $\boldsymbol{z} = [\boldsymbol{z}_n \quad \boldsymbol{z}_m]^T$, $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_n \quad \boldsymbol{\Phi}_m]$, with $\boldsymbol{z}_n \in \mathbb{C}^{n\times 1}$, $\boldsymbol{z}_m \in \mathbb{C}^{m\times 1}$, and using (27), we arrive at two subsystems of (29):

$$\begin{aligned} \dot{\boldsymbol{z}}_n(t) &= \boldsymbol{J}_n\boldsymbol{z}_n(t) \ ; \\ \boldsymbol{H}_m\dot{\boldsymbol{z}}_m(t) &= \boldsymbol{z}_m(t) \ . \end{aligned}$$

The first subsystem has solution:

$$\boldsymbol{z}_n(t) = e^{\boldsymbol{J}_n t}\boldsymbol{z}_n(0) \ . \tag{30}$$

For the second subsystem let m_* be the index of the nilpotent matrix $\boldsymbol{H}_m$, i.e. $\boldsymbol{H}_m^{m_*} = \boldsymbol{0}_{m,m}$. Then we can obtain the following matrix equations:

$$\begin{aligned} \boldsymbol{H}_m\dot{\boldsymbol{z}}_m(t) &= \boldsymbol{z}_m(t) \\ \boldsymbol{H}_m^2\ddot{\boldsymbol{z}}_m(t) &= \boldsymbol{H}_m\dot{\boldsymbol{z}}_m(t) \\ &\vdots \\ \boldsymbol{H}_m^{m_*-1}\boldsymbol{z}_m^{(m_*-1)}(t) &= \boldsymbol{H}_m^{m_*-2}\boldsymbol{z}_m^{(m_*-2)}(t) \\ \boldsymbol{H}_m^{m_*}\boldsymbol{z}_m^{(m_*)}(t) &= \boldsymbol{H}_m^{m_*-1}\boldsymbol{z}_m^{(m_*-1)}(t) \ . \end{aligned}$$

By taking the sum of the above equations we arrive at the following solution for the second subsystem:

$$\boldsymbol{z}_m(t) = \boldsymbol{0}_{m,1} \ . \tag{31}$$

By using (31) in (28), we obtain:

$$\boldsymbol{\xi}(t) = \begin{bmatrix} \boldsymbol{\Phi}_n & \boldsymbol{\Phi}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_n(t) \\ \boldsymbol{0}_{m,1} \end{bmatrix} = \boldsymbol{\Phi}_n\boldsymbol{z}_n(t) \ . \tag{32}$$

Substitution of (30) in the last equation yields:

$$\boldsymbol{\xi}(t) = \boldsymbol{\Phi}_n e^{\boldsymbol{J}_n t}\boldsymbol{z}_n(0) \ . \tag{33}$$

The matrix $\boldsymbol{J}_n$ has the Jordan canonical form. In addition, $e^{\boldsymbol{J}_n t}$ is the matrix exponential of $\boldsymbol{J}_n t$.

Let $\boldsymbol{\Psi}_n$, $\boldsymbol{\Psi}_m$ be the matrices that contain all left eigenvectors of the finite, and infinite eigenvalues respectively. Then by setting $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_n \quad \boldsymbol{\Psi}_m]^T$, and making use of (27) we have that $\boldsymbol{\Psi}_n\boldsymbol{B}\boldsymbol{\Phi}_n = \boldsymbol{I}_n$. By multiplying (32) with $\boldsymbol{\Psi}_n\boldsymbol{B}$ we have:

$$\boldsymbol{\Psi}_n\boldsymbol{B}\boldsymbol{\xi}(t) = \boldsymbol{\Psi}_n\boldsymbol{B}\boldsymbol{\Phi}_n\boldsymbol{z}_n(t) \ ,$$

or, equivalently, $\boldsymbol{z}_n(t) = \boldsymbol{\Psi}_n\boldsymbol{B}\boldsymbol{\xi}(t)$. Hence,

$$\boldsymbol{z}_n(0) = \boldsymbol{\Psi}_n\boldsymbol{B}\boldsymbol{\xi}(0) \ .$$

If we replace the above expression in the general solution (33) we have:

$$\boldsymbol{\xi}(t) = \boldsymbol{\Phi}_n e^{\boldsymbol{J}_n t}\boldsymbol{\Psi}_n\boldsymbol{B}\boldsymbol{\xi}(0) \ .$$

If in the above equation we substitute:

$$\begin{aligned} \boldsymbol{\Phi}_n &= [\boldsymbol{\phi}_1^{(n_1)} \dots \ \boldsymbol{\phi}_1^{(1)} \ \dots \ \boldsymbol{\phi}_\nu^{(n_v)} \dots \ \boldsymbol{\phi}_\nu^{(1)}] \ , \\ \boldsymbol{\Psi}_n &= [\boldsymbol{\psi}_1^{(n_1)} \dots \ \boldsymbol{\psi}_1^{(1)} \ \dots \ \boldsymbol{\psi}_\nu^{(n_v)} \dots \ \boldsymbol{\psi}_\nu^{(1)}]^T \ , \end{aligned}$$

we arrive at (16).

The proof is completed.

Proof of Proposition 1. From the general solution (16), the evolution of $\xi_k(t)$ is:

$$\xi_k(t) = \sum_{i=1}^{\nu} e^{\lambda_i t} \sum_{j=1}^{n_i} \big(\sum_{\sigma=1}^{j} t^{\sigma-1} \boldsymbol{\psi}_i^{(j-\sigma+1)} \boldsymbol{B}\boldsymbol{\xi}(0) \big) \phi_{k,i}^{(j)} \ , \tag{34}$$

where ${\phi_{k,i}}^{(j)} \in \boldsymbol{\phi}_i^{(j)}$. Partial differentiation of (34) with respect to $e^{\lambda_i t}$ leads to:

$$\frac{\partial \xi_k(t)}{\partial e^{\lambda_i t}} = \sum_{j=1}^{n_i} \big(\sum_{\sigma=1}^{j} t^{\sigma-1} \boldsymbol{\psi}_i^{(j-\sigma+1)} \boldsymbol{B}\boldsymbol{\xi}(0) \big) \phi_{k,i}^{(j)} \ . \tag{35}$$

We are interested in calculating the PF at $t \to 0$. Substitution in (35) yields:

$$\frac{\partial \xi_k(t)}{\partial e^{\lambda_i t}} = \boldsymbol{\psi}_i^{(1)} \boldsymbol{B} \boldsymbol{\xi}(0) \phi_{k,i}^{(1)} \ . \tag{36}$$

The PF of ξ_k in the finite mode λ_i is given by (36) by applying appropriate initial conditions, i.e. $\xi_k(0) = 1$, $\xi_h(0) = 0$, $h \neq k$ [1]. Since $k \leq n$, we have $\boldsymbol{\psi}_i^{(1)} \boldsymbol{B} \boldsymbol{\xi}(0) = \psi_{i,k}^{(1)}$. Substitution in (36) gives:

$$\frac{\partial \xi_k(t)}{\partial e^{\lambda_i t}} = \boldsymbol{\psi}_i^{(1)} \boldsymbol{B} \boldsymbol{\xi}(0) \phi_{k,i}^{(1)} = \psi_{i,k}^{(1)} \phi_{k,i}^{(1)} \ = p_{k,i}. \tag{37}$$

In the special case that the eigenvectors form a complete basis for the rational vector space of the matrix pencil, i.e. all eigenvalues are either distinct or their algebraic multiplicity is equal with the geometric, we have $n_i = 1$, $\nu = n$, and thus $\phi_{i,k}^{(1)} = \phi_{i,k}$, $\psi_{i,k}^{(1)} = \psi_{i,k}$ in (37).

The proof is completed.

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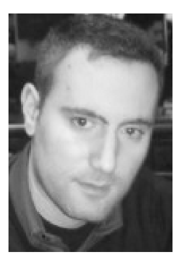

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