A Systematic Method to Model Power Systems as Stochastic Differential Algebraic Equations

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***Abstract*—This paper proposes a systematic and general approach to model power systems as continuous stochastic differential-algebraic equations. With this aim, the paper provides a theoretical background on stochastic differential-algebraic equations and justifies the need for stochastic models in power system analysis. Then, the paper describes a general procedure to define stochastic dynamic models. Practical issues related to the numerical integration of the resulting power system model are also discussed. A case study illustrating the proposed approach is provided based on the IEEE 145-bus 50-machine system. The case study also illustrates and compares the reliability of the results obtained using stochastic and conventional probabilistic models.**

***Index Terms*—Stochastic differential algebraic equations (SDAE), power system dynamics, time domain integration, Wiener's process, Ornstein-Uhlenbeck's process.**

I. Introduction

ANY physical system and, thus, also power systems, contains randomness and uncertainty. For example, load power consumption is not fully deterministic. Moreover, in recent years, the massive installation of non-dispatchable technologies, e.g., wind parks, has increased the degree of randomness in power systems. While stochastic programming has been extensively taken into account in power system economics and operation, e.g., [1], the modeling of continuous stochastic processes has not been yet assessed on a methodological basis. With this aim, this paper provides a novel and systematic approach aimed to model any power system device in terms of stochastic differential-algebraic equations.

A. Literature Review on SDE and SDAE

The literature on Stochastic Differential Equations (SDEs) is vast. Theoretical background on SDEs can be found in, e.g., [2]–[6]. SDEs are widely applied in finance to model stochastic fluctuations of stock prices and other financial assets [7], [8], and in several fields of science and engineering to study physical systems affected by different stochastic phenomena [9]–[12].

In general, SDEs can only be solved through numerical methods. References [13], [14] provide detailed descriptions of the available fixed step size methods for the numerical solution of SDEs, whereas variable step size methods for SDEs have been proposed in [15] and [16].

The authors are supported by the Ministry of Education of Spain, Project ENE2012-31326.

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Power system variables evolve in different time scales. To take into account this behavior, power systems are traditionally modeled as a set of Differential-Algebraic Equations (DAEs). Due to the stiffness of this model, implicit numerical methods must be used in simulations to avoid numerical instability. On the other hand, if stochastic differential equations are used to model random perturbations in power systems, the system model becomes a set of Stochastic Differential-Algebraic Equations (SDAEs). Therefore, solving SDAE models involves to deal with both stochastic terms and stiffness. With this regard, in [17], the adequacy of different implicit fixed step size numerical methods for SDAEs is discussed. In the context of electronic circuit simulation, [18] shows that implicit numerical methods with fixed step size used to solve SDEs are also suitable for being applied to SDAEs.

B. Application of SDEs to Power System Analysis

The application of SDEs to topics related to power systems is limited. Traditionally, the focus has been on modeling load behaviors [19]–[22]. In [23] SDEs are used as a planning tool for power systems. In particular, SDEs are used to model small perturbations in both system loads and transmission line parameters. A similar approach is used in [24] and [25] to analyze power system dynamics, where discrete perturbations are included into switching events due to the operation of tap-changing transformers. Power system voltage stability is studied in [26]–[29] where SDEs are used to model the load behavior. In [30], both load and wind power production are modeled with SDEs to address the problem of power system balance management in an hourly time frame. More recently, in [31] random loads are modeled through SDEs which are included directly in the algebraic equations of a power system model. The problems related to the appearance of singularities in the model resulting from this approach are investigated in [32]. Finally, stochastic transient stability is discussed in [33], and the application of SDEs to wind speed modeling is analyzed in [34], [35].

C. Contributions

Despite stochastic models have been considered in the last decades and have been periodically revisited in the literature, we consider that a systematic approach to define SDAE models is still missing. We identify the following major limitations of the models given in the literature.

1) All models proposed in the literature are rigidly formulated in terms of a given stochastic process (e.g., the Markov's jump process in [21] and the Ornstein-Uhlenbeck's process in [27]).

2) The intrinsic difficulty in including stochastic perturbations into algebraic variables or parameters leads to arbitrary solutions such as the singular perturbation approach used, for example, in [20] and [22].
3) Time domain simulations are not given (e.g., [23]–[25]), or limited to small power systems. For example, the WSCC 9-bus system is used in recent publications such as [27] and [33]).

This paper attempts to address the limitations above. With this aim, the main objects of the paper are to provide a general algorithm-based tool to include stochastic processes in power system models in the form of general-formulated SDEs; and to properly simulate the effect of stochastic terms on the transients of power systems of any size. The proposed technique can be applied to systems of any order and complexity as it is a general mathematical approach that does not require any assumption or particular property to be satisfied by the original system.

In particular, the paper provides:

1) A quantitative tool for evaluating the weight of stochastic perturbations on the power system transient behavior.
2) A systematic yet simple approach to include stochastic terms in power system models.
3) A mathematical-based comparison between the proposed SDAE-based approach and other approaches proposed in the literature. In particular we consider the pure deterministic model and the so-called probabilistic model.
4) A case study that illustrates the points above.

D. Paper Organization

The remainder of the paper is organized as follows. Section II briefly outlines SDE. Section III describes the power system model based on SDAE and formulates the initial value problem for deterministic, probabilistic and stochastic algebraic differential equations. Section IV presents the proposed systematic approach to model electric power systems in terms of SDAEs. Section V discusses issues related to the numerical integration of SDAEs. Section VI presents some examples of stochastic power system models. This section also compares the initial value problems defined in Section III through simulations based on the IEEE 145-bus 50-machine system. Finally, Section VII draws conclusions and proposes future work.

II. Outline of Stochastic Differential Equations

A multi-dimensional Stochastic Differential Equation (SDE) has the following general form:

$$d\boldsymbol{\eta}(t) = \boldsymbol{a}(\boldsymbol{\eta}, t)dt + \boldsymbol{b}(\boldsymbol{\eta}, t)d\boldsymbol{w}(t) \tag{1}$$

where $\boldsymbol{a}$ ($\boldsymbol{a} : \mathbb{R}^{n_\eta} \times \mathbb{R}^+ \mapsto \mathbb{R}^{n_\eta}$) is the *drift* of the SDE, $\boldsymbol{b}$ ($\boldsymbol{b} : \mathbb{R}^{n_\eta} \times \mathbb{R}^+ \mapsto \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_{\rm w}}$) is an $n_\eta \times n_{\rm w}$ tensor that represents the *diffusion* of the SDE, and $\boldsymbol{w}$ is a $n_{\rm w}$-dimensional vector composed of $n_{\rm w}$ independent scalar Wiener's processes.

The Wiener's process has various formal definitions. The most practical is likely the following one: a real-valued continuous-time stochastic process $w(t)$, $t \in [0, +\infty)$ is a Wiener's process if:

1) $w(0) = 0$.
2) The function $t \mapsto w(t)$ is almost surely continuous.
3) The function $w(t)$ is of unbounded variation in every interval.
4) $w(t)$ has independent increments with $w(t+h) - w(t) \sim \mathcal{N}(0, h)\ \forall t, h > 0$.

where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2. The fourth condition means that if $t_1 \le t_2 \le \cdots \le t_{j-1} \le t_j$, then $w(t_2) - w(t_1), \dots, w(t_j) - w(t_{j-1})$ are independent random variables. The expectation of the Wiener's process is $E(w(t)) = 0$ and the variance is:

$$E(w^2(t)) - E^2(w(t)) = t \tag{2}$$

The results for the expectation and variance follow immediately from the definition that increments have a normal distribution centered at zero, thus $w(t) = w(t) - w(0) \sim \mathcal{N}(0, t)$.

Another property of the Wiener's process is that it is not differentiable, which means that $\lim_{\Delta t \to 0}(w(t + \Delta t) - w(t))/\Delta t$ does not exist. However, there does exist a formal mathematical definition

$$\xi(t) = \frac{dw(t)}{dt} \tag{3}$$

which is the so-called *white noise* process. Observe that (3) is just a formal way to relate the concept of Wiener's process with the concept of white noise. To simplify the notation, in the remainder of the paper, we implicitly assume the time dependence of ξ.

Because Wiener's process is not differentiable, the correct mathematical formulation of (1) is actually the integral form

$$\boldsymbol{\eta} = \boldsymbol{\eta}(t_0) + \int_{t_0}^{t} \boldsymbol{a}(\boldsymbol{\eta}, s)ds + \int_{t_0}^{t} \boldsymbol{b}(\boldsymbol{\eta}, s)d\boldsymbol{w}(s), \quad t \in [t_0, t_f], \tag{4}$$

where the first integral is an ordinary Riemann-Stieltjes' integral and the second one is a stochastic integral. Due to the unbounded variation of the Wiener's process, stochastic integrals cannot be interpreted as Riemann-Stieltjes' integrals. With this regard, there are mainly three different interpretations of stochastic integrals: the Itô's and the Stratonovich's approaches, and the backward integral. The details of these interpretations are beyond the scope of this paper. The interested reader can find further insights and a wider literature review in [17]. In this paper, we use the Itô's interpretation for the stochastic integrals and, for simplifying the notation, we use the differential representation of SDEs. In particular, from equation (1) and according to (3) the general formulation used for SDEs is

$$\dot{\boldsymbol{\eta}} = \boldsymbol{a}(\boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{\eta})\boldsymbol{\xi} \tag{5}$$

where, for simplicity, the dependence on time of the terms of the equation has been omitted.

In the general case, SDEs cannot be explicitly solved and numerical methods are needed. Numerical methods for SDEs can show two types of convergence: *strong* and *weak*.

Strong convergence refers to the goodness of the approximation when the focus is on the process trajectories themselves, and is a straightforward generalization of the usual convergence criterion applied to the numerical schemes for DAEs.

Formally, an approximation $\boldsymbol{\eta}_N$ converges strongly with order $\beta_{\rm S} > 0$ to the solution $\boldsymbol{\eta}$ at time t_N if there exist a positive constant c, independent of Δt, such that

$$||\boldsymbol{\eta}(t_N) - \boldsymbol{\eta}_N|| \leq c(\Delta t)^{\beta_{\rm S}} \tag{6}$$

with $\Delta t \in (0, \overline{\Delta t})$, and $\overline{\Delta t} > 0$ is given step length.

On the other hand, weak convergence refers to the goodness of the approximation of the statistical properties of the solutions to the statistical properties of the process. Formally, an approximation $\boldsymbol{\eta}_N$ converges weakly with order $\beta_{\rm W} > 0$ to the solution $\boldsymbol{\eta}$ at time t_N if there exists a positive constant c, independent of Δt, such that

$$||E(\mathcal{M}(\boldsymbol{\eta}(t_N)) - E(\mathcal{M}(\boldsymbol{\eta}_N))|| \leq c(\Delta t)^{\beta_{\rm W}} \tag{7}$$

with $\Delta t \in (0, \overline{\Delta t})$, where $\mathcal{M}$ is a smooth function satisfying certain polynomial growth conditions [13], that usually represents a moment.

Reference [13] provides a detailed description of the available methods for the numerical solution of SDEs.

III. Modeling Power Systems as SDAEs

The transient behavior of electric power systems is traditionally described through a set of differential algebraic equations (DAE) as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, t) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, t) \end{aligned} \tag{8}$$

where $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \times \mathbb{R}^{n_{\rm u}} \times \mathbb{R}^{+} \mapsto \mathbb{R}^{n_{\rm x}}$) are the differential equations, $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \times \mathbb{R}^{n_{\rm u}} \times \mathbb{R}^{+} \mapsto \mathbb{R}^{n_{\rm y}}$) are the algebraic equations, $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathbb{R}^{n_{\rm x}}$) are the state variables, e.g., rotor speeds and rotor angles of synchronous machines, the dynamic states of loads and system controllers, etc., $\boldsymbol{y}$ ($\boldsymbol{y} \in \mathbb{R}^{n_{\rm y}}$) are the algebraic variables, e.g., bus voltage magnitudes and phases, and $\boldsymbol{u}$ ($\boldsymbol{u} \in \mathbb{R}^{n_{\rm u}}$) are discrete variables modeling events, e.g., line outages and faults, switching operation of tap-changers, etc., and $t \in \mathbb{R}^{+}$ is the time.

In common practice, equations (8) are split into a collection of subsystems where discrete variables $\boldsymbol{u}$ are substituted for *if-then* rules. Thus, (8) can be conveniently rewritten as a finite collection of continuous DAEs, one per each discrete variable change. Such a system is also known as *hybrid automaton* or *hybrid dynamical system*. An in-depth description and formalization of hybrid systems for power system analysis can be found in [36], [37]. Thus, without loss of generality, in the remainder of the paper, we focus only on autonomous and continuous DAEs, as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \end{aligned} \tag{9}$$

where $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \mapsto \mathbb{R}^{n_x}$), and $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \mapsto \mathbb{R}^{n_{\rm y}}$).

Despite the fact that (9) are well-accepted and are the common choice in power system software packages, some aspects of the reality are missing from this formulation, e.g., stochastic behavior and variable functional relations. In this paper, we are interested in defining the possible effects of stochastic perturbations on the transient behavior of (9). This kind of perturbations can be originated by the stochastic variations of loads, transient rotor vibrations of synchronous machines, harmonics, EMT transients, measurement errors of control devices, etc. The effect of such perturbations can lead to stochastic behaviors of the main system variables, e.g., frequency, voltages, and power flows. In the general case, these stochastic processes can depend on power system variables and parameters. Therefore, and based on (5), stochastic perturbations are modeled as follows:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta})\boldsymbol{\xi} \tag{10}$$

where $\boldsymbol{a} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \times \mathbb{R}^{n_{\eta}} \mapsto \mathbb{R}^{n_{\eta}}$, and $\boldsymbol{b} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \times \mathbb{R}^{n_{\eta}} \mapsto \mathbb{R}^{n_{\eta}} \times \mathbb{R}^{n_{\rm w}}$.

By introducing (10) in (9) the DAEs are transform into a set of Stochastic Differential Algebraic Equations (SDAEs), as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}) \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta})\boldsymbol{\xi} \end{aligned} \tag{11}$$

where functions $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \times \mathbb{R}^{n_{\eta}} \times \mathbb{R}^{n_{\eta}} \mapsto \mathbb{R}^{n_{\rm x}}$) and $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{n_{\rm x}} \times \mathbb{R}^{n_{\rm y}} \times \mathbb{R}^{n_{\eta}} \mapsto \mathbb{R}^{n_{\rm y}}$) are modified to include the effect of stochastic terms $\boldsymbol{\eta}$. Observe that, in (11), $\boldsymbol{g}$ does not explicitly depend on white noises $\boldsymbol{\xi}$ nor on $\dot{\boldsymbol{\eta}}$, which allows solving (11) by means of state-of-art integration techniques for SDAEs, [13].

A. Stochastic Initial Value Problem versus Deterministic and Probabilistic Ones

In this subsection, we compare the initial value problems for SDAE, DAE and Probabilistic DAE (PDAE). Equations (11) can be used to define a stochastic initial value problem (SIVP), which is formally obtained by (11) and the initial conditions:

$$\begin{aligned} \boldsymbol{x}(t_0) &= \boldsymbol{x}_0 \\ \boldsymbol{y}(t_0) &= \boldsymbol{y}_0 \\ \boldsymbol{\eta}(t_0) &= \boldsymbol{\eta}_0 \end{aligned} \tag{12}$$

For the sake of unifying the notation, we rewrite the SIVP (11)-(12) in the following compact expression:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}, \dot{\boldsymbol{\eta}}), & \boldsymbol{x}(t_0) &= \boldsymbol{x}_0 \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}), & \boldsymbol{y}(t_0) &= \boldsymbol{y}_0 \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta})\boldsymbol{\xi}, & \boldsymbol{\eta}(t_0) &= \boldsymbol{\eta}_0 \end{aligned} \tag{13}$$

Preserving the structure of (13), the deterministic initial value problem (DIVP) is:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{0}, \boldsymbol{0}), & \boldsymbol{x}(t_0) &= \boldsymbol{x}_0 \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{0}), & \boldsymbol{y}(t_0) &= \boldsymbol{y}_0 \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{0}, & \boldsymbol{\eta}(t_0) &= \boldsymbol{0} \end{aligned} \tag{14}$$

Finally, the following probabilistic (or random) initial value problem (PIVP) has been largely used in power system analysis [38]–[40]:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}_0, \boldsymbol{0}), & \boldsymbol{x}(t_0) &= \boldsymbol{x}_0 \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\eta}_0), & \boldsymbol{y}(t_0) &= \boldsymbol{y}_0 \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{0}, & \boldsymbol{\eta}(t_0) &= \boldsymbol{\eta}_0 \end{aligned} \tag{15}$$

where the effect of the stochastic variables $\boldsymbol{\eta}$ is just due to their initial values $\boldsymbol{\eta}_0$, but for $t > t_0$ the evolution of the system is purely deterministic. Hence the trajectories of (14) and (15) are regulated by the same differential equations and only differ because of the initial condition.

It is relevant to note that (14) and (15) are particular cases of the general formulation given in (13).

IV. Modeling Stochastic Perturbations in Power Systems

In this section, we present a general approach to model the stochastic behavior of power systems. With this aim, stochastic inputs are considered *perturbations* and stochastic models are derived by introducing stochastic variations, as discussed in the previous section. For the sake of simplicity, but without lack of generality, the discussion is restricted to a one-dimensional model. Let z be a system variable or parameter and $\psi(z,t)$ a function representing its deterministic trajectory. A stochastic model for this trajectory can be derived by introducing a stochastic variation $\eta(z,t,\xi)$ as follows:

$$z(t) = \psi(z,t) + \eta(z,t,\xi) \tag{16}$$

Equation (16) can be particularized as follows:

1) If z represents a state variable ($z \equiv x$), the trajectory function $\psi(x,t)$ is not generally known a priori, but it is the solution of a differential equation. Therefore, by taking derivatives with respect to time in both sides of equation (16) we have

$$\frac{dx(t)}{dt} = f(x,t) + \frac{d\eta(x,t,\xi)}{dt} \tag{17}$$

where

$$f(x,t) = \frac{d\psi(x,t)}{dt} \tag{18}$$

The dynamic behavior of the stochastic variable $\eta(x,t,\xi)$ is an SDE

$$\dot{\eta} = \frac{d\eta(x,t,\xi)}{dt} = a(x,\eta,t) + b(x,\eta,t)\xi \tag{19}$$

where ξ is defined in (3). Therefore, (17) and (19) can be rewritten as the following set of SDEs:

$$\begin{aligned} \dot{x} &= f(x,t) + \dot{\eta} \\ \dot{\eta} &= a(x,\eta,t) + b(x,\eta,t)\xi \end{aligned} \tag{20}$$

2) The variable z can represent an algebraic variable, e.g., $z \equiv y$. Taking into account that this kind of variable is constrained by an algebraic equation, e.g., $g(y,t) = 0$, its deterministic trajectory $\psi(y,t)$ can be obtained by formally imposing that:

$$y(t) = g^{-1}(t) \tag{21}$$

where g^{-1} is the inverse of $g(y,t)$. By combining (21) in (16), one obtains:

$$y(t) = g^{-1}(t) + \eta(y,t,\xi) \tag{22}$$

Therefore, the stochastic model for an algebraic variable y is a SDAE where the stochastic perturbation η imposes that y varies according to the constraint g, as follows:

$$\begin{aligned} 0 &= g(y-\eta,t) \\ \dot{\eta} &= a(y,\eta,t) + b(y,\eta,t)\xi \ . \end{aligned} \tag{23}$$

3) In case z represents a constant system parameter, say p ($z \equiv p$), its deterministic trajectory is a constant, e.g., $\psi(p,t) = p_0$. Therefore, according to (16), the stochastic model for a parameter is as follows:

$$\begin{aligned} p(t) &= p_0 + \eta(p,t,\xi) \\ \dot{\eta} &= a(p,\eta,t) + b(p,\eta,t)\xi \ . \end{aligned} \tag{24}$$

4) The multidimensional case can be derived straightforwardly by simply defining as many sets of (20), (23), and (24), as needed. Correlations between stochastic processes can be easily taken into account through a proper definition of the elements of $\boldsymbol{a}$ and $\boldsymbol{b}$.

It is important to note that functions a and b of the SDEs in (20), (23), and (24) define the dynamic behavior and the statistical properties, respectively, of the stochastic perturbations η. Therefore, taking into account the characteristics of the stochastic phenomena perturbing the system, an appropriate SDE can be designed.

The formulation proposed in this section allows to straightforwardly derive SDAEs from DAEs by simply including additional (stochastic) differential equations to the original DAE model. In particular, in the case study presented in Section VI, we use (20), (23), and (24) to introduce stochastic variations in synchronous machine rotor speeds, bus voltage phasors, and voltage dependent loads, respectively.

V. Time-Domain Integration of SDAEs

The most common methods used for integrating SDEs and SDAEs are explicit, being the most popular ones the Euler and Milstein schemes [14]. These methods have been also used for studying power system transients, e.g., [33]. On the other hand, explicit schemes are particularly exposed to numerical errors, which tend to increase as the simulation time increases. Moreover, the stiffness of power system equations makes explicit integration schemes particularly prone to numerical issues.

In this paper, we consider an implicit trapezoidal integration scheme for the drift (deterministic) functions, $\boldsymbol{f}$ and $\boldsymbol{a}$, and an explicit Maryuama-Euler scheme for the diffusion (stochastic) term $\boldsymbol{b}$ [14]. For simplifying the notation of the integration scheme, let us define $\hat{\boldsymbol{x}} = [\boldsymbol{x}^T, \boldsymbol{\eta}^T]^T$, $\hat{\boldsymbol{f}} = [\boldsymbol{f}^T, \boldsymbol{a}^T]^T$, and $\hat{\boldsymbol{b}} = [\boldsymbol{0}^T, \boldsymbol{b}^T]^T$, which transform (11) into the following compact form:

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \hat{\boldsymbol{f}}(\hat{\boldsymbol{x}}, \boldsymbol{y}) + \hat{\boldsymbol{b}}(\hat{\boldsymbol{x}}, \boldsymbol{y})\boldsymbol{\xi} \\ \boldsymbol{0} &= \boldsymbol{g}(\hat{\boldsymbol{x}}, \boldsymbol{y}) \end{aligned} \tag{25}$$

The i-th step of the integration scheme applied to (25) is as follows:

$$\begin{aligned}\hat{\boldsymbol{x}}^{(i)} &= \hat{\boldsymbol{x}}^{(i-1)} + \frac{1}{2}\left(\hat{\boldsymbol{f}}(\hat{\boldsymbol{x}}^{(i)}, \boldsymbol{y}^{(i)}) + \hat{\boldsymbol{f}}^{(i-1)}\right)\Delta t \\ &\quad + \hat{\boldsymbol{b}}^{(i-1)}\Delta \boldsymbol{w}^{(i-1)} \\ \boldsymbol{0} &= \boldsymbol{g}(\hat{\boldsymbol{x}}^{(i)}, \boldsymbol{y}^{(i)})\end{aligned} \tag{26}$$

where $\Delta t = t_i - t_{i-1}$ is the time step length, $\hat{\boldsymbol{x}}^{(i-1)}$, $\hat{\boldsymbol{f}}^{(i-1)}$, and $\hat{\boldsymbol{b}}^{(i-1)}$ are known vectors from the previous step, and $\Delta \boldsymbol{w}^{(i-1)} = \boldsymbol{w}(t_i) - \boldsymbol{w}(t_{i-1})$ is the vector of increments of the Wiener's process. This scheme provides a strong convergence order of 0.5 ($O(\Delta t)^{0.5}$) and a weak convergence order of 1 ($O(\Delta t)$), [13], [14].

When considering a numerical solution of SDAEs, one has to restrict the attention to a finite subinterval $[t_0, t_f]$. Moreover, it is necessary to choose an appropriate discretization $t_0 < t_1 < \cdots < t_i < \cdots < t_N = t_f$ of $[t_0, t_f]$. The other crucial problem is simulating sample paths of the Wiener's process over the discretization of $[t_0, t_f]$. Considering an equally-spaced discretization $t_j - t_{j-1} = (t_f - t_0)/N = h$, $j = 1, \ldots, N$, the random independent increments are:

$$w(t_j) - w(t_{j-1}) \sim \mathcal{N}(0, h), \qquad n = 1, \ldots, N \tag{27}$$

of the Wiener's process $\{w(t), t_0 \leq t \leq t_f\}$. From (27) is clear that the values of the Wiener's process increments depend on the size of h and, therefore, the use of a different step size h will lead to a different path of the Wiener's process.

The general approach is to use the same step size in the numerical integration and in the generation of the Wiener's process, i.e., $\Delta t = h$. If the focus of the analysis is on the trajectories themselves and, for example, comparing trajectories of the solutions of SDAEs with different discretization steps Δt is of interest, it is necessary to ensure that the same path of the Wiener's process is being used in order to make the comparison consistent. To solve this issue, the sample paths of the Wiener's process can be generated by using a step size h equal to the smallest discretization step Δt used in the integration scheme.

VI. Case Study

This section presents some results of time domain simulations for a power system including stochastic perturbations in a variety of devices as specified in Subsection VI-A.

Since there is no commercial tool able to define and integrate power system device models with inclusion of stochastic processes, simulations can be solved only in two ways: (i) to adapt an existing open-source software tool for power system analysis to account for SDAE models (e.g., PSAT [41]) or (ii) to adapt an existing SDE software tool to study power system models (e.g., SDE Toolbox [42]). We have inclined towards the former option. With this aim, we have used DOME [43], a Python-based version of PSAT, which allows easily prototyping novel integration methods and device models such as the ones discussed in this paper. Moreover, DOME allows parallelizing time domain simulations, which dramatically speeds up the simulation of SDAEs. The sampling of normal variates that approximates the Wiener's process is achieved by computer generation of pseudo-random numbers. In particular, we have used the pseudo-random number generator provided by the GNU Scientific Library (GSL) [44].

A. Stochastic Processes considered in the Case Study

In this case study, we consider stochastic perturbations of load power consumption, bus voltage phasors, and synchronous machine rotor speeds. The choice of these variables and parameters is driven by the purpose of illustrating the procedure discussed in Section IV, which can be applied to state variables (i.e., rotor speeds), algebraic variables (i.e., bus voltage phasors) as well as system parameters (e.g., load power consumption). Clearly, any other stochastic perturbation can be included by using the proposed approach and based on the knowledge of the system and on measurements.

For the stochastic processes involved in the simulations, we use the Ornstein-Uhlenbeck's process, also known as *mean-reverting* process. This process has been already used in the literature for modeling power system loads (see, for example, [27]–[29]). The general form of a SDE defining the Ornstein-Uhlenbeck's process is:

$$\dot{\eta}(t) = \alpha(\mu - \eta(t)) + b\xi \tag{28}$$

Equation (28) represents an exponentially autocorrelated process that tends to a pre-specified mean value μ as time tends to infinite. The resulting process is a normal distributed process whose expression for the mean and the variance are:

$$\mathrm{E}[\eta(t)] = \mu + (\eta(t_0) - \mu)e^{-\alpha t} \tag{29}$$

$$\mathrm{Var}[\eta(t)] = \frac{b^2}{2\alpha}(1 - e^{-2\alpha t}) \tag{30}$$

Hence, $\eta(t)$ tends to a normal distributed process in the form $\mathcal{N}(\mu, b^2/2\alpha)$ as $t \to \infty$. Note that α is the mean-reversion speed, that is, the rate at which the stochastic variable is pulled toward the mean value μ, and it can be used to define the autocorrelation of the process. After that, b can be adjusted to obtain the desired variance.

In general, the Ornstein-Uhlenbeck's process is more appropriate for modeling stochastic perturbations in physical systems than the standard Wiener's process, as the variance of the Ornstein-Uhlenbeck's process does not increase indefinitely. However, observe that other stochastic processes with different statistical properties can be used, as the procedure described in Section IV is general and does not rely on a specific stochastic process. In particular, one can choose other functions for the drift and the diffusion terms of the SDE (19) to define stochastic processes with given statistical properties. For example, the interested reader can find a discussion on how to model Weibull's distributed processes as SDAEs in [35].

1) Load model: Loads at high voltage level are equivalents of wide areas containing several thousands of physical loads and devices. The stochastic behavior of such equivalent aggregated loads is a well-established concept that has been recognized in the literature (e.g., [19]).

The stochastic load model is developed based on the well-known voltage dependent load model as a reference [45]:

$$\begin{aligned} p_{\mathrm{L}}(t) &= p_{\mathrm{L}0}(v(t)/v_0)^{\gamma} \\ q_{\mathrm{L}}(t) &= q_{\mathrm{L}0}(v(t)/v_0)^{\gamma} \end{aligned} \tag{31}$$

where $p_{\mathrm{L}0}$ and $q_{\mathrm{L}0}$ are parameters representing active and reactive load powers at $t = 0$, $v(t)$ is the voltage magnitude at the bus where the load is connected, v_0 is the value of this voltage magnitude at $t = 0$, and exponent γ is a parameter that characterizes the dependence of the load with respect to voltage. For instance, this exponent takes values $\gamma = 0$ for constant power loads, $\gamma = 1$ for constant current loads, and $\gamma = 2$ for constant impedance loads. Since $p_{\mathrm{L}0}$ and $q_{\mathrm{L}0}$ are parameters, equation (24) of the procedure described in Section IV applies. Parameters $p_{\mathrm{L}0}$ and $q_{\mathrm{L}0}$ in (31) can be transformed into stochastic processes $\eta_p(t)$ and $\eta_q(t)$, respectively, with the same formulation as (28). The resulting set of equations is thus:

$$\begin{aligned} p_{\mathrm{L}}(t) &= (p_{\mathrm{L}0} + \eta_p(t))(v(t)/v_0)^{\gamma} \\ q_{\mathrm{L}}(t) &= (q_{\mathrm{L}0} + \eta_q(t))(v(t)/v_0)^{\gamma} \\ \dot{\eta}_p(t) &= \alpha_p(\mu_p - \eta_p(t)) + b_p\xi_p \\ \dot{\eta}_q(t) &= \alpha_q(\mu_q - \eta_q(t)) + b_q\xi_q \end{aligned} \tag{32}$$

where the parameters α, μ, and b, have the same meaning as in (28).

2) Synchronous machine rotor speeds: The dynamic of a synchronous machine is described by the well-known equations [45]:

$$\begin{aligned} \dot{\delta}(t) &= \omega(t) - \omega_{\mathrm{s}} \\ \dot{\omega}(t) &= \frac{1}{M}(\tau_m(t) - \tau_e(t) - D(\omega(t) - \omega_{\mathrm{s}})) \end{aligned} \tag{33}$$

where δ is the rotor angle, ω is the rotor speed, τ_m is the mechanical torque, τ_e is the electro-magnetic torque, M is the inertia constant, D is the rotor damping, and ω_{s} is the synchronous speed. Expressions of τ_e and τ_m depend on the level of detail of the model and controllers associated with the machine [45]. The stochastic perturbations of rotor speeds model rotor vibrations due to electrical harmonics as well as mechanical asymmetry and aging. The stochastic process perturbs the state variable ω. Hence, equation (20) of the procedure described in Section IV applies. By introducing the Ornstein-Uhlenbeck's process (28) in (33), one obtains:

$$\begin{aligned} \dot{\delta}(t) &= \omega(t) - \omega_{\mathrm{s}} \\ \dot{\omega}(t) &= \frac{1}{M}(\tau_m(t) - \tau_e(t) - D(\omega(t) - \omega_{\mathrm{s}})) + \dot{\eta}_\omega \\ \dot{\eta}_\omega(t) &= \alpha_\omega(\mu_\omega - \eta_\omega(t)) + b_\omega\xi_\omega \end{aligned} \tag{34}$$

where the parameters α, μ, and b, have the same meaning as in (28). Observe that classical stochastic models of synchronous machines add just a diffusion term to (33), e.g., [20]. However, in this particular case, the resulting model reproduces an Ornstein-Uhlenbeck's process due to the damping term in (33).

3) Bus voltage phasors: To complete the set of stochastic models, we consider the bus voltage phasors, which, in transient stability studies, are algebraic variables [45]. As for synchronous machine rotor speeds, stochastic processes of bus voltage phasors model harmonics due to transformer nonlinearities and power electronic devices, the effects of electromagnetical transients, corona effects, and any other parasitic phenomena that result, at the fundamental frequency, as noise.

Voltage phasors have to satisfy the power (or current) balances at all buses, which are a subset of the algebraic equations $\boldsymbol{g}$ in (9). The power balance at each bus n is:

$$\begin{aligned} 0 &= p_{\mathrm{G}n}(t) - p_{\mathrm{L}n}(t) \\ &\quad - v_n(t)\sum_{m=1}^{n_{\mathrm{B}}}[v_m(t)B_{nm}\sin(\theta_n(t) - \theta_m(t)) \\ &\quad + v_m(t)G_{nm}\cos(\theta_n(t) - \theta_m(t))] \\ 0 &= q_{\mathrm{G}n}(t) - q_{\mathrm{L}n}(t) \\ &\quad - v_n(t)\sum_{m=1}^{n_{\mathrm{B}}}[v_m(t)G_{nm}\sin(\theta_n(t) - \theta_m(t)) \\ &\quad - v_m(t)B_{nm}\cos(\theta_n(t) - \theta_m(t))] \end{aligned} \tag{35}$$

where n_{B} is the number of buses connected to bus n, $p_{\mathrm{G}n}$ and $q_{\mathrm{G}n}$ are, respectively, the total active and reactive powers generated at bus n, $p_{\mathrm{L}n}$ and $q_{\mathrm{L}n}$ are, respectively, the total active and reactive powers consumed at bus n, G_{nm} and B_{nm} are, respectively, the real a imaginary part of the element $\{n, m\}$ of the system admittance matrix, v_n (v_m) is the voltage magnitude at bus n (m), and θ_n (θ_m) is the voltage phase angle at bus n (m).

Since (35) are algebraic, we apply equation (23) of the procedure described in Section IV. Introducing the Ornstein-Uhlenbeck's process, one obtains the following set of SDAE for each node of the system:

$$\begin{aligned} 0 &= p_{\mathrm{G}n}(t) - p_{\mathrm{L}n}(t) \\ &\quad - \hat{v}_n(t)\sum_{m=1}^{n_{\mathrm{B}}}[\hat{v}_m(t)B_{nm}\sin(\hat{\theta}_n(t) - \hat{\theta}_m(t)) \\ &\quad + \hat{v}_m(t)G_{nm}\cos(\hat{\theta}_n(t) - \hat{\theta}_m(t))] \\ 0 &= q_{\mathrm{G}n}(t) - q_{\mathrm{L}n}(t) \\ &\quad - \hat{v}_n(t)\sum_{m=1}^{n_{\mathrm{B}}}[\hat{v}_m(t)G_{nm}\sin(\hat{\theta}_n(t) - \hat{\theta}_m(t)) \\ &\quad - \hat{v}_m(t)B_{nm}\cos(\hat{\theta}_n(t) - \hat{\theta}_m(t))] \\ \dot{\eta}_{v_n}(t) &= \alpha_{v_n}(\mu_{v_n} - \eta_{v_n}(t)) + b_{v_n}\xi_{v_n} \\ \dot{\eta}_{\theta_n}(t) &= \alpha_{\theta_n}(\mu_{\theta_n} - \eta_{\theta_n}(t)) + b_{\theta_n}\xi_{\theta_n} \end{aligned} \tag{36}$$

where $\hat{v}_n(t) = v_n(t) - \eta_{v_n}(t)$ and $\hat{\theta}_n(t) = \theta_n(t) - \eta_{\theta_n}(t)$ ($\hat{v}_m(t)$ and $\hat{\theta}_m(t)$ are defined in a similar way), and parameters α, μ and b, have the same meaning as in (28).

B. IEEE 145-bus 50-machine System

In this subsection, we provide simulation results of the initial value problem for SDAEs through the IEEE 145-bus 50-machine system [46]. This system consists of 145 buses, 453 line/transformers, and 50 machines. Machines connected to buses 93, 102, 104, 105, 106, 110, and 111, are modeled through a VI-order model. These machines are equipped with IEEE ST1a exciters including PSS devices. The classical model is used for the remaining machines. Turbine governors have been incorporated to all system machines to avoid unrealistic instabilities. The case study includes stochastic perturbations for all loads, rotor speed of synchronous machines, and bus voltage phasors by using the models described in the previous subsection. Besides stochastic processes, at $t = 1$ s, the system undergoes the outage of the line connecting buses 6 and 7.

The sample paths of the Wiener's process used in the stochastic model are generated by using a step size $h = 0.02$ s, whereas a step length $\Delta t = 0.02$ is used for the integration scheme. 1000 simulations with different Wiener's processes are carried out to provide a consistent set of solutions. The final simulation time is 20 s. The parameters for the stochastic models are as follows: $\alpha_p = \alpha_q = \alpha_\omega = \alpha_v = \alpha_\theta = 0.5$ 1/s, $b_p = 0.5\%$ of $p_{\rm L0}$, $b_q = 0.5\%$ of $q_{\rm L0}$, $b_\omega = 0.01\%$ of ω_0, $b_v = 0.01\%$ of v_0, and $b_\theta = 0.01\%$ of θ_0, where $p_{\rm L0}$, $q_{\rm L0}$, ω_0, v_0 and θ_0 are the values of the parameters and variables obtained from the initialization of the corresponding models from the power flow solution. Moreover, we assume $\mu_p = \mu_q = \mu_\omega = \mu_v = \mu_\theta = 0$ to impose that the stochastic perturbations have zero mean. Finally, the initial values of the stochastic perturbations are taken from a $\mathcal{N}(0, b^2/2\alpha)$ particularized for the given values of b and α corresponding to each perturbation model.

Figure 1 depicts the time evolution of the voltage magnitude at bus 95 for the 1000 simulations. The black continuous line is the voltage mean value, which coincides with the solution of the deterministic initial value problem, i.e., with the solution of a system where stochastic perturbations are not modeled. Figure 1 shows that the deterministic solution is above the minimum voltage technical limit. However, 38.4% of the solutions of the SDAE system are below such limit. A similar information can be obtained using the probabilistic model. As stated in Section III-A, in the probabilistic model only initial conditions are affected by random perturbations, whereas the system dynamic model is purely deterministic. These random perturbations on the initial conditions are taken from a $\mathcal{N}(0, b^2/2\alpha)$ particularized for the given values of b and α corresponding to each perturbation model used in the stochastic simulation described above. Figure 2 shows 1000 trajectories obtained using probabilistic model of loads, machine rotor speeds and bus voltage phasors. In this case, 20.4% of the solutions falls below the bus voltage limit. Hence, the probabilistic model provides conservative results with respect to the detailed SDAE model. This result was to be expected as SDAE models accurately account for the time evolution, i.e., the "history" of stochastic perturbations. Table I summarizes simulation results.

By using a server mounting 48 CPUs, 256 GB of RAM, and running a 64-bit Linux OS, the simulation of a single trajectory of the SDAE model takes an average CPU time of 19.8 s, whereas the parallel simulation of 1000 trajectories on the 48 CPUs takes an average total time of 551 s.

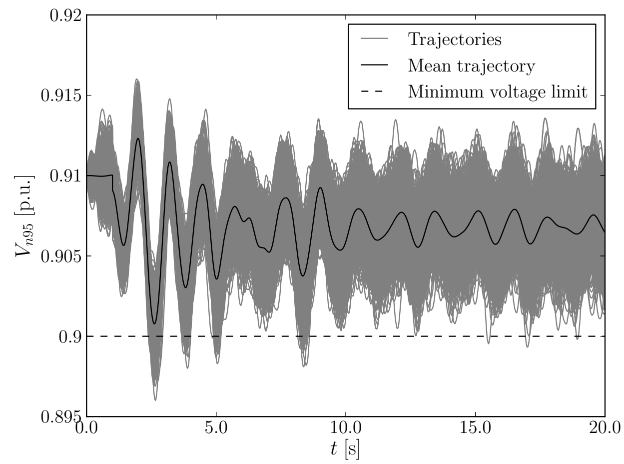

Fig. 1. Bus voltage magnitude at bus 95 obtained by using the stochastic approach.

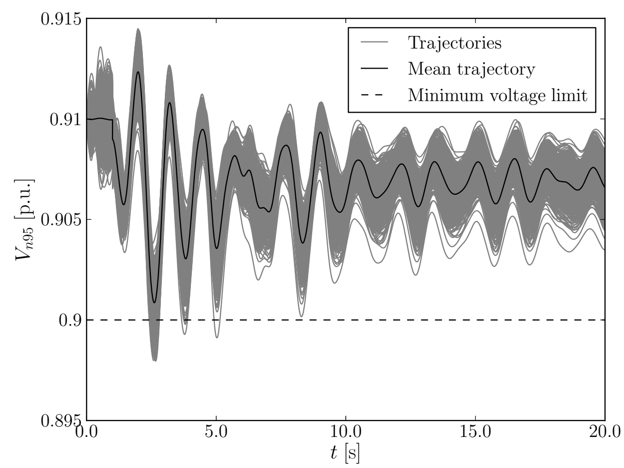

Fig. 2. Bus voltage magnitude at bus 95 obtained by using the probabilistic approach.

VII. Conclusions and Future Works

In this paper, we propose a general formulation for modeling power systems as stochastic differential algebraic equations, as well as a systematic method to model the stochastic perturbations present in these systems. The convenience of the stochastic modeling approach is illustrated through simulations on the IEEE 145-bus 50-machine system.

The proposed methodology has the relevant advantage of being general and can easily account for any stochastic process formulated by means of stochastic differential equations. Based on this methodology, several future work directions can be anticipated. Detailed modeling of non-dispatchable distributed generation (e.g., photo-voltaic and wind parks) is one promising field of future research. Moreover, the tools provided in this paper can be used to study transient stability as well as long term voltage stability of power systems affected by stochastic inputs. Small-signal stability analysis is also a

TABLE I
SIMULATION RESULTS FOR THE 145-BUS, 50-MACHINE SYSTEM

Problem	LVP	NTLV #	PTLV %
DIVP	No	0	0
PIVP	Yes	204	20.4
SIVP	Yes	384	38.4

LVP: Low Voltage Phenomenon
NTLV: Number of Trajectories with Low Voltage
PTLV: Percentage of Trajectories with Low Voltage

challenge for SDAE systems. The analythical tool provided in this paper can be also useful to define metrics able to help system operators quantify the effects of stochastic processes. The authors are currently working on all these topics.

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